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ALGEBRAIC COMPUTATIONS OF SCALED PADE FRACTIONS

by

© Dong-Koo Choi

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled "Algebraic Computations of Scaled Pade Fractions", submitted by Dong-Koo Choi in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

Two companion algorithms are developed for constructing Pade fractions along an off-diagonal path of the Pade table for a function $-A(z) / B(z)$, where $A(z)$ and $B(z)$ are formal power series over a field.

One of the algorithms computes the first n Pade fractions along the off-diagonal in time $O(n^2)$. When $A(z)$ and $B(z)$ are finite power series (i.e., polynomials), it is shown that the algorithm is equivalent to Euclid's extended algorithm for computing greatest common divisors.

The other algorithm, a generalization of the first, proceeds along the off-diagonal in quadratic steps, and is of complexity $O(n \log^2 n)$. When $A(z)$ and $B(z)$ are polynomials, the second algorithm becomes a fast Euclid's extended algorithm for computing greatest common divisors. The algorithm is of the same complexity as other fast greatest common divisor methods, but its iterative nature provides a practical advantage during implementation.

The algorithms may also be used for computing Pade fractions along an anti-diagonal path of the Pade table, as well. The fast algorithm is of the same complexity as other fast algorithms for anti-diagonal computations. However, it has the advantage of being able to determine easily any specific Pade fraction along the anti-diagonal.

Finally, it is shown that two successive Pade fractions can be used to obtain the inverse of Hankel and Toeplitz matrices in time $O(n \log^2 n)$.

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CHAPTER 1

INTRODUCTION

The Pade table of a fomal power series

$$A(z) = \sum_{i=0}^{\infty} a_i z^i \tag{1.1}$$

is a doubly infinite array of rational functions

$$\frac{U_{mn}(z)}{V_{mn}(z)} = \frac{\sum_{i=0}^{m} u_i z^i}{\sum_{i=0}^{n} v_i z^i} \tag{1.2}$$

determined in such a manner that the Maclaurin expansion of $U_{mn}(z)/V_{mn}(z)$ agrees with $A(z)$ as far as possible. The power series $A(z)$ is said to be normal if , for each pair (m,n), this agreement is exact through the power z^{m+n}. The foundation for the development of Pade theory was laid by Cauchy(1821) in his famous "Cour d'Analyse". Later, Frobenius(1881) developed the basic algorithmic aspects of the theory, and Pade(1892) treated in detail certain abnormal cases.

Since Pade's time, Pade tables have become a classical tool of analysis. Their analytical properties have been studied in great depth and are surveyed, for example, by Gragg [*GRA*74] and by Baker [*BAK*75]. Traditionally, it is assumed that the coefficients in (1.1) and (1.2) lie in the field of complex numbers, and that the power series and the rational functions are to be evaluated at certain points in the complex plane.

Although the results obtained in this thesis are likely to have an impact in an analytical (or numerical) setting, the effects of this impact are not examined. The issues addressed are strictly algebraic ones; that is, no consideration is given to the goodness of the approximation of (1.2) to (1.1). Instead, the objective is to provide an effective tool to algebraically manipulate rational func-

tions as truncated power series (for which the cost of operations is relatively cheap), and to transform back to rational form on request. It is assumed that the coefficients lie in an arbitrary field.

The coefficients of the rational function (1.2) satisfy an under-determined system of linear equations, known as the Hankel system. Various properties of the family of solutions to the Hankel sytem are described in Chapter 2. All the results given are those of Pade, and are included for the sake of completeness and for reference ease. The Hankel system can be solved directly for the coefficients of $U_{mn}(z)$ and $V_{mn}(z)$. With coefficients over a field, Rissannen [*RIS*73], for example, provides an algorithm which requires $O(n^2)$ arithmetic operations over the field. On the other hand, with coefficients over an arbitrary integral domain, Geddes [*GED*79] gives a fraction-free algorithm which requires $O(n^3)$ arithmetic operations over the integral domain.

Various relationships are known to exist between neighboring elements in the Pade table. These relationships have been used to derive numerous $O(n^2)$ methods for computing a sequence of elements in the Pade table. A survey and comparison of these methods are given in Brezinski [*BRZ*76], Claessens [*CLA*75] and Wynn [*WYN*60]. All these methods have a major flaw; they may fail in the abnormal case. In Chapter 3, a new relationship between elements lying along an off-diagonal path in the Pade table is derived. This leads to yet another $O(n^2)$ method, however, the new method succeeds in the abnormal case. Furthermore, if the coefficient field has an appropriate n-th root of unity (which permits fast multiplication and division of polynomials), the asymptotic complexity of the algorithm becomes $O(n \log^2 n)$.

The new algorithm can be applied to the quotient of two power series. Then, in particular, it can be applied to the quotient of two finite power series (i.e. the quotient of two polynomials). In Chapter 4, it is shown that if all elements along a specific off-diagonal of the Pade table are computed, then the new algorithm is equivalent to Euclid's extended algorithm for computing greatest common divisors. Furthermore, if fast polynomial operations can be performed, the new

algorithm can compute the greatest common divisor of two polynomials in $O(n \log^2 n)$ arithmetic operations. The algorithm has three advantages over the other fast methods (Moenck [MOE73], Aho et al[AHO74], and Brent et al [BRE80]) for computing greatest common divisors. It is basically an iterative algorithm rather than a recursive one, and consequently, significant cost savings can result during implementation. Secondly, it produces intermediate polynomial remainder sequences as a by-product, which is a valuable feature for some applications. Finally, various details about the nature of its behavior are easier to comprehend.

The algorithm can be applied to the quotient of the reciprocals of two truncated power series (polynomials). It is shown in Section 4.4 that this yields successive elements along an anti-diagonal path of the Pade table.

In Chapter 5, it is shown that two successive elements along the diagonal path of the Pade table can be used to find the inverse of a Hankel matrix. Therefore, if fast polynomial operations are possible, the inverse of a Hankel matrix of order n can be determined in $O(n \log^2 n)$ arithmetic operations. The new inversion method handles abnormalities with greater ease, and less cost, than does Brent et al's algorithm [BRE80]. Furthermore, by proceeding along an anti-diagonal path, the algorithm can be used to compute the inverse of a Toeplitz matrix of order n in time $O(n \log^2 n)$.

CHAPTER 2

PADE THEORY

2.1. Introduction

This chapter examines the theoretical background of Pade theory. None of the results are new; however, some of the proofs may be original. All of the proofs are obtained directly from the properties of Hankel systems.

The highlight of this chapter is Corollary 2.6, due to Pade, which completely describes the family of solutions to the Hankel system. We define one specific member of this family to be a scaled Pade fraction. Scaled Pade fractions exist uniquely, and are fundamental to the development of subsequent algorithms.

2.2. Pade Forms

2.2.1. Definitions

The class **P** of formal power series over a field **F** consists of expressions of the form

$$A(z) = \sum_{i=0}^{\infty} a_i z^i$$

with coefficients $a_i \in \mathbf{F}$. We denote the units of **P** by

$$\mathbf{U} = \{ A(z) = \sum_{i=0}^{\infty} a_i z^i \mid a_0 \neq 0 \, , A(z) \in \mathbf{P} \} \, .$$

Associated with each such unit is a set of rational functions defined as follows:

Definition: Let $A(z) \in \mathbf{U}$, and let m and n be non-negative integers. The rational form

$$\frac{U_{mn}(z)}{V_{mn}(z)} = \frac{u_0 + u_1 z + \cdots + u_m z^m}{v_0 + v_1 z + \cdots + v_n z^n} \tag{2.1}$$

is called a **Pade form** of type (m, n) for $A(z)$ if

(a) $V_{mn}(z) \neq 0$, and (2.2)

(b) $A(z) \cdot V_{mn}(z) - U_{mn}(z) = O(z^{m+n+1})$. (2.3)

The (algebraic) O-symbol indicates that the right side is a power series beginning with the power $z^{m+n+k+1}$, $0 \leq k \leq \infty$; $k = +\infty$ means that $A(z) \cdot V_{mn}(z) - U_{mn}(z) = 0$.

From an analytical point of view, this means that the rational form $U_{mn}(z) / V_{mn}(z)$ is determined so that its Maclaurin expansion agrees with $A(z)$ as far as possible. That is, equation (2.3) corresponds to a linear system of $m+n+1$ equations in the $m+n+2$ unknowns $u_0, \cdots, u_m, v_0, \cdots, v_n$ as follows:[1]

$$\sum_{j=0}^{n} a_{i-j} \cdot v_j = \begin{cases} u_i, & i = 0, 1, \cdots, m, \\ 0, & i = m+1, \cdots, m+n . \end{cases} \tag{2.4}$$

Equivalently, (2.4) can be written as the following subsystems for the polynomials $U_{mn}(z)$ and $V_{mn}(z)$:[2]

$$\begin{bmatrix} u_m \\ \cdot \\ \cdot \\ \cdot \\ u_0 \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} \tag{2.5}$$

and

---

[1] By convention, $a_i = 0$ if $i < 0$.

[2] Throughout this thesis, the notation $P(z) = p_0 + p_1 z + \cdots + p_k z^k$ for a polynomial P, shall be used interchangeably with the vector notation $P = (p_k, p_{k-1}, \cdots, p_0)^t$ for its coefficients.

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{2.6}$$

The matrix

$$H_{mn} = \begin{bmatrix} a_{m-n+1} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n-1} \end{bmatrix} \tag{2.7}$$

associated with the power series $A(z)$ is called a **Hankel matrix**. With this definition, the system (2.6) becomes

$$H_{mn} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_1 \end{bmatrix} = -v_0 \cdot \begin{bmatrix} a_{m+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{m+n} \end{bmatrix}. \tag{2.8}$$

Denote the **determinant** of H_{mn} by $\det(H_{mn})$, and let $\det(H_{m,0}) = 1$.

2.2.2. Existence and Non-uniqueness

Theorem 2.1 (Frobenius): There always exists a non-trivial solution of the systems (2.5) and (2.6). Equivalently, Pade forms of type (m, n) for $A(z) \in \mathbf{U}$ always exist.

Proof: The existence of $V_{mn}^t = (v_n, \cdots, v_0)$ satisfying (2.6) follows immediately from the fact that the $n+1$ columns of the matrix in (2.6) are vectors of length n and must therefore be linearly dependent. The vector $U_{mn}^t = (u_m, \cdots, u_0)$ satisfying (2.5) is then obtained simply by multiplication. ■

It is clear that if U'_{mn} and V'_{mn} is a solution of (2.5) and (2.6), then so is $\alpha \cdot U'_{mn}$ and $\alpha \cdot V'_{mn}$, where α is a non-zero constant. The non-uniqueness of the solution, however, is more profound than this, and the next few results identify the true nature of the family of solutions.

Lemma 2.2: Let $U_{mn}(z)/V_{mn}(z)$ be a Pade form of type (m, n) for $A(z) \in \mathbf{U}$, and let $D(z)$ be a non-zero polynomial of degree $\partial(D) \leq \lambda$, where $\lambda = \min\{m - \partial(U_{mn}), n - \partial(V_{mn})\}$. Then $U'_{mn}(z)/V'_{mn}(z)$, where

$$U'_{mn}(z) = U_{mn}(z)D(z) \quad \text{and} \quad V'_{mn}(z) = V_{mn}(z)D(z) \ ,$$

is also a Pade form of type (m, n) for $A(z)$.

Proof: With the $U_{mn}(z)$ and $V_{mn}(z)$ defined above, equation (2.5) becomes

$$\begin{bmatrix} 0_\lambda \\ u_{m-\lambda} \\ \vdots \\ u_0 \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda} \\ \vdots \\ v_0 \end{bmatrix}, \tag{2.9}$$

where 0_λ refers to the zero vector of length λ. Multiplication by the polynomial D then results in

$$\begin{bmatrix} d_0 & \cdots d_\lambda & & 0 \\ & & \ddots & d_\lambda \\ & \ddots & & \vdots \\ 0 & & & d_0 \end{bmatrix} \begin{bmatrix} 0_\lambda \\ u_{m-\lambda} \\ \vdots \\ u_0 \end{bmatrix}$$

$$= \begin{bmatrix} d_0 & \cdots d_\lambda & & 0 \\ & & \ddots & d_\lambda \\ & \ddots & & \vdots \\ 0 & & & d_0 \end{bmatrix} \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda} \\ \vdots \\ v_0 \end{bmatrix}$$

$$
= \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} d_0 & \cdots & d_\lambda & & \\ & \ddots & & \ddots & \\ & & \ddots & & d_\lambda \\ & & & \ddots & \vdots \\ & \mathbf{0} & & & d_0 \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda} \\ \vdots \\ v_0 \end{bmatrix} .
$$

That is,

$$
\begin{bmatrix} u'_m \\ \cdot \\ \cdot \\ \cdot \\ u'_0 \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} v'_n \\ \cdot \\ \cdot \\ \cdot \\ v'_0 \end{bmatrix} .
$$

Therefore, U'_{mn} and V'_{mn} satisfy equation (2.5). The fact that V'_{mn} also satisfies (2.6) follows from

$$
\begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} v'_n \\ \cdot \\ \cdot \\ \cdot \\ v'_0 \end{bmatrix}
$$

$$
= \begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} d_0 & \cdots & d_\lambda & & \\ & \ddots & & \ddots & \\ & & \ddots & & d_\lambda \\ & & & \ddots & \vdots \\ & \mathbf{0} & & & d_0 \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda} \\ \vdots \\ v_0 \end{bmatrix}
$$

$$
= \begin{bmatrix} d_\lambda & \cdots & d_0 & & & \\ & \ddots & & \ddots & & 0 \\ & & \ddots & & \ddots & \\ & \mathbf{0} & & d_\lambda & \cdots & d_0 \end{bmatrix} \begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} v_{n-\lambda} \\ \vdots \\ v_0 \\ 0_\lambda \end{bmatrix}
$$

$$
= \begin{bmatrix} d_\lambda & \cdots & d_0 & & & \\ & \ddots & & \ddots & & 0 \\ & & \ddots & & \ddots & \\ & \mathbf{0} & & d_\lambda & \cdots & d_0 \end{bmatrix} \begin{bmatrix} a_{m-n-\lambda+1} & \cdots & a_{m-\lambda+1} \\ \cdot & & \cdot \\ a_{m-n} & \cdots & a_m \\ a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n-\lambda} \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda} \\ \vdots \\ v_0 \end{bmatrix}
$$

$$= \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \tag{2.10}$$

In the last equality of (2.10), use was made both of equation (2.9) and of equation (2.6) with $V'_{mn} = (0_\lambda, v_{n-\lambda}, \cdots, v_0)$. ■

The converse of Lemma 2.2 is given by

Lemma 2.3. Let $U'_{mn}(z)/V'_{mn}(z)$ be a Pade form of type (m, n) for $A(z) \in \mathbf{U}$, and let $D(z)$, $D(0) \neq 0$, be a common divisor of U'_{mn} and V'_{mn}. Then $U_{mn}(z)/V_{mn}(z)$, where

$$U_{mn}(z) = U'_{mn}(z)/\, D(z) \quad \text{and} \quad V_{mn}(z) = V'_{mn}(z)/\, D(z) \ ,$$

is also a Pade form of type (m, n) for $A(z)$.

Proof: The arguments for the proof are similar to those of the proof of Lemma 2.2. ◘

The condition that $D(0) \neq 0$ in Lemma 2.3 is necessary as can be seen from the following simple example.

Example 2.1: Let $A(z) = 1 + z + z^2 + 2z^3 + 3z^4 + 4z^5 + 5z^6 \cdots$.

Clearly, $U'_{2,4} = z$ and $V'_{2,4} = z - z^2 - z^4$ yield a Pade form of type (2,4) since

$$A(z) \cdot V'_{2,4}(z) - U'_{2,4}(z) = O(z^7) \, .$$

A common divisor of $U'_{2,4}(z)$ and $V'_{2,4}(z)$ is $D(z) = z$. However, the rational function $U(z)/V(z)$, where

$$U(z) = U'_{2,4}(z)/\, D(z) = 1 \ \text{ and } \ V(z) = V'_{2,4}(z)/\, D(z) = 1 - z - z^3,$$

does not yield a Pade form of type (2,4) since

$$A(z)\cdot V(z) - U(z) = O(z^6). \quad \blacksquare$$

2.2.3. General Structure of Pade Forms

Let H_{mn} be given. If $\det(H_{mn}) \neq 0$, then system (2.8) exhibits a one-parameter family of solutions. Indeed, if $\det(H_{mn}) \neq 0$ and $v_0 = 1$ in (2.8), then the solution of (2.8) for $V_{mn}(z)$ and of (2.5) for $U_{mn}(z)$ is unique. Pade forms for which $V(0) = 1$, i.e., $v_0 = 1$ are said to be **normalized.** In addition, we have

Lemma 2.4. If $\det(H_{mn}) \neq 0$, then the solutions U_{mn} and V_{mn} of (2.5) and (2.6) are relatively prime.

Proof: Suppose the contrary, and let $D(z)$ be a non-trivial common divisor of $U_{mn}(z)$ and $V_{mn}(z)$.

If $D(0) = 0$, then $V_{mn}(0) = 0$, i.e., $v_0 = 0$. Consequently, (2.8) has only the trivial solution and this contradicts the definition of a Pade form.

If $D(0) \neq 0$, let $\bar{D}(z) = D(z)/D(0)$. Then, by Lemma 2.3, $U'(z) = U(z)/\bar{D}(z)$ and $V'(z) = V(z)/\bar{D}(z)$ are solutions to systems (2.5) and (2.6). But $V'(0) = V(0)$, and equation (2.8) together with the condition that $\det(H_{mn}) \neq 0$ then imply that $V'(z) = V(z)$. Thus, $D(z)$ must be a polynomial of degree zero, which contradicts the initial assumption that $D(z)$ is non-trivial. $\blacksquare$

The converse of Lemma 2.4, however, is not true, as can be seen in the following example.

Example 2.2: Consider again the power series $A(z)$ in example 2.1. It can be verified that a Pade form of type (4, 3) for $A(z)$ is $U_{4,3}(z)/V_{4,3}(z) = (1 - z + z^3) / (1 - 2z + z^2)$, where $U_{4,3}(z)$ and $V_{4,3}(z)$ are relatively prime. However, the Hankel matrix

$$H_{4,3} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$$

associated with the Pade form of type (4, 3) is singular, so that $\det(H_{4,3})=0$. $\blacksquare$

Given the Hankel matrix H_{mn}, let $H_{m-\lambda,n-\lambda}$ be its largest non-singular principal submatrix. That is, λ is the smallest non-negative integer for which $\det(H_{m-\lambda,n-\lambda}) \neq 0$. From Lemma 2.4, a Pade form

$$\frac{U^{*}_{m-\lambda,n-\lambda}(z)}{V^{*}_{m-\lambda,n-\lambda}(z)} = \frac{\sum_{i=0}^{m-\lambda} u_i^{*} z^i}{\sum_{i=0}^{n-\lambda} v_i^{*} z^i} \tag{2.11}$$

of type ($m-\lambda, n-\lambda$) for $A(z)$ exists, where $U^{*}_{m-\lambda,n-\lambda}(z)$ and $V^{*}_{m-\lambda,n-\lambda}(z)$ are relatively prime and $v_0^{*} \neq 0$. The Pade form is unique up to a multiplicative constant and satisfies

$$\begin{bmatrix} u^{*}_{m-\lambda} \\ \cdot \\ \cdot \\ \cdot \\ u_0^{*} \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_{m-\lambda} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n+\lambda} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} v^{*}_{n-\lambda} \\ \cdot \\ \cdot \\ \cdot \\ v_0^{*} \end{bmatrix} \tag{2.12}$$

and

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_{m-\lambda+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m-\lambda} & \cdots & a_{m+n-2\lambda} \end{bmatrix} \begin{bmatrix} v^{*}_{n-\lambda} \\ \cdot \\ \cdot \\ \cdot \\ v_0^{*} \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{2.13}$$

In addition,

$$A(z) \cdot V^{*}_{m-\lambda,n-\lambda}(z) - U^{*}_{m-\lambda,n-\lambda}(z) = z^{m-\lambda+n-\lambda+k+1} \sum_{i=0}^{\infty} r_i z^i, \tag{2.14}$$

where $r_0 \neq 0$, if $k < \infty$.

Theorem 2.5: Let λ be such that $H_{m-\lambda,n-\lambda}$ is the largest non-singular principal submatrix in H_{mn} , and let k be determined by (2.14). Then $k \geq \lambda$. In addition, let $l = \min\{2\lambda, k\}$. Then a basis for the solution space of

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{2.15}$$

is given by the $l-\lambda+1$ vectors

$$\{(v^*_{n-\lambda}, \cdots, v^*_0, 0_\lambda)^t, \cdots, (0_{l-\lambda}, v^*_{n-\lambda}, \cdots, v^*_0, 0_{2\lambda-l})^t\}.$$

Proof: If $\lambda=0$, then the theorem follows trivially from Lemma 2.4. If $\lambda \neq 0$, then the multiplication of (2.15) by

$$\begin{array}{c} n-\lambda\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \\ \lambda\left\{\begin{array}{c} \\ \\ \\ \\ \end{array}\right. \end{array} \begin{bmatrix} 1 & & & & & \\ & \ddots & & & \mathbf{0} & \\ 0 & & & & & \\ & & & 1 & & \\ v^*_{n-\lambda} & \cdots & & v^*_0 & & \\ & \ddots & & & \ddots & \\ 0 & & & v^*_{n-\lambda} & \cdots & v^*_0 \end{bmatrix}$$

yields

$$\begin{bmatrix} a_{m-n+1} & & \cdots & & a_{m+1} \\ \cdot & & & & \vdots \\ \cdot & & & & \\ a_{m-\lambda} & & \cdots & & a_{m+n-\lambda} \\ & & & r_0 \;\cdots & r_{\lambda-k} \\ & & \cdot^{\cdot^{\cdot}} & & \vdots \\ \mathbf{0} & r_0 & & \cdots & r_{2\lambda-k-1} \end{bmatrix} \begin{bmatrix} v_n \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ 0 \end{bmatrix} \tag{2.16}$$

in the case that $k < \lambda$, or

$$\begin{bmatrix} a_{m-n+1} & \cdots & & a_m & a_{m+1} \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & \cdot \\ a_{m-\lambda} & \cdots & & \cdot & a_{m+n-\lambda} \\ & & & & 0_{l-\lambda} \\ & & & & r_0 \\ & \mathbf{0} & & r_0 & r_1 \\ & & \iddots & & \vdots \\ & & r_0 & \cdots & r_{2\lambda-l-1} \end{bmatrix} \begin{bmatrix} v_n \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ \cdot \\ \cdot \\ \cdot \\ \\ \\ 0 \end{bmatrix} \tag{2.17}$$

in the case that $k \geq \lambda$. But the largest non-singular principal submatrix in (2.16) has order n, which violates the assumption that $\lambda \neq 0$. Thus, when $\lambda \neq 0$, only (2.17) is possible, which proves that $k \geq \lambda$.

From (2.17), it follows that the coeffcient matrix of (2.17) has rank $n + \lambda - l$, and therefore the solution space of (2.15) is of dimension $l - \lambda + 1$. But, using (2.13) and (2.14), it is easy to see that all the vectors

$$\{(\dot{v}_{n-\lambda}, \cdots, \dot{v}_0, 0_\lambda)^t, \cdots, (0_{l-\lambda}, \dot{v}_{n-\lambda}, \cdots, \dot{v}_0, 0_{2\lambda-l})^t\}$$

satisfy (2.17). ■

Corollary 2.6 (Pade): The general solution of

$$\begin{bmatrix} u_m \\ \cdot \\ \cdot \\ \cdot \\ u_0 \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix}$$

and

$$\begin{bmatrix} a_{m-n+1} & \cdots & a_{m+1} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_m & \cdots & a_{m+n} \end{bmatrix} \begin{bmatrix} v_n \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

is given by

$$U_{mn}(z) = z^{2\lambda-l}\left(\sum_{i=0}^{l-\lambda}\alpha_i z^i\right)\left(\sum_{i=0}^{m-\lambda}u_i^* z^i\right) \tag{2.18}$$

$$V_{mn}(z) = z^{2\lambda-l}\left(\sum_{i=0}^{l-\lambda}\alpha_i z^i\right)\left(\sum_{i=0}^{n-\lambda}v_i^* z^i\right), \tag{2.19}$$

where l and λ are defined in Theorem 2.5 and α_i, $0 \le i \le l-\lambda$, is arbitrary.

Proof: Linear combinations of the basis vectors for the solution space can be written as

$$V_{mn} = \alpha_{l-\lambda}\begin{bmatrix} v_{n-\lambda}^* \\ \cdot \\ \cdot \\ \cdot \\ v_0^* \\ 0_\lambda \end{bmatrix} + \cdots + \alpha_0 \begin{bmatrix} 0_{l-\lambda} \\ v_{n-\lambda}^* \\ \vdots \\ v_0^* \\ 0_{2\lambda-l} \end{bmatrix}$$

$$= \begin{bmatrix} 0_\lambda & \alpha_0 & \cdots & \alpha_{l-\lambda} & \mathbf{0} \\ & & \ddots & \ddots & \alpha_{l-\lambda} \\ & & & & \\ & \mathbf{0} & & & \alpha_0 \\ & & & & 0_{2\lambda-l} \end{bmatrix} \begin{bmatrix} 0_\lambda \\ v_{n-\lambda}^* \\ \vdots \\ v_0^* \end{bmatrix},$$

which in polynomial form becomes (2.19). Furthermore, using equation (2.12) and the results of Theorem 2.5, it follows that

$$\begin{bmatrix} 0_i \\ u_{m-\lambda}^* \\ \vdots \\ u_0^* \\ 0_{\lambda-i} \end{bmatrix} = \begin{bmatrix} a_{m-n} & \cdots & a_m \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{-n} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} 0_i \\ v_{n-\lambda}^* \\ \vdots \\ v_0^* \\ 0_{\lambda-i} \end{bmatrix}$$

for $0 \le i \le l-\lambda$. Thus, the polynomial form for $U_{mn}(z)$ corresponding to $V_{mn}(z)$ is given by (2.18). ∎

2.3. Pade Fractions

2.3.1. Scaled Pade Fractions

Let H_{mn} for $A(z) \in \mathbf{U}$ be given, and let $H_{m-\lambda,n-\lambda}$ be its largest non-singular principal submatrix. If the rational function

$$\frac{U^{*}_{m-\lambda,n-\lambda}(z)}{V^{*}_{m-\lambda,n-\lambda}(z)} = \frac{\sum_{i=0}^{m-\lambda} u_i^{*} z^i}{\sum_{i=0}^{n-\lambda} v_i^{*} z^i} \tag{2.20}$$

is the Pade form of type $(m-\lambda, n-\lambda)$ for $A(z)$, then

Definition: The **scaled Pade fraction** of type (m,n) for $A(z)$ is defined to be $\gamma_{mn}(z) = S_{mn}(z) \,/\, T_{mn}(z)$, where

$$S_{mn}(z) = z^{\lambda} \sum_{i=0}^{m-\lambda} u_i^{*} z^i \tag{2.21}$$

$$T_{mn}(z) = z^{\lambda} \sum_{i=0}^{n-\lambda} v_i^{*} z^i \ . \tag{2.22}$$

Theorem 2.7: The scaled Pade fraction γ_{mn} of type (m,n) for $A(z)$ is a Pade form of type (m,n) for $A(z)$. Furthermore, γ_{mn} is unique up to a multiplicative constant.

Proof: The scaled Pade fraction is obtained simply by setting $\alpha_{l-\lambda} \neq 0$ and $\alpha_0 = \alpha_1 = \cdots = \alpha_{l-\lambda-1} = 0$ in equations (2.18) and (2.19) of Corollary 2.6. ■

As an immediate consequence, we have

Corollary 2.8. $\det(H_{mn}) \neq 0$ if and only if $T_{mn}(0) \neq 0$.

An alternative definition of scaled Pade fractions is given by

Theorem 2.9. $S_{mn}(z)/T_{mn}(z)$, where $T_{mn}(z) \neq 0$ is the scaled Pade fraction of type (m,n) for $A(z)$ if

(a) $\min\{\, m - \partial(S_{mn})\,,\, n - \partial(T_{mn})\,\} = 0$,

(b) $GCD(\, S_{mn}\,,\, T_{mn}\,) = z^{\lambda_{mn}}$, for some $\lambda_{mn} \geq 0$, and

(c) $A(z) \cdot T_{mn}(z) - S_{mn}(z) = O(\, z^{m+n+1}\,)$.

Proof: The result is an immediate consequence of Theorem 2.7. ∎

Thus, when normalized, the scaled Pade fraction of type (m,n) for $A(z)$ exists uniquely, and in addition satisfies the systems (2.5) and (2.6). This is tremendous advantage for our purposes over other definitions of Pade fractions described below, since it simplifies the development of algorithms in subsequent chapters.

2.3.2. Non-scaled Pade Fractions

Let $U_{mn}(z) \,/\, V_{mn}(z)$ be a Pade form of type (m,n) for $A(z)$, and let

$$D(z) = GCD(U_{mn}\,,\, V_{mn}) \,.$$

Gragg [*GRA* 72] defines the Pade fraction of type (m,n) for $A(z)$ to be $P_{mn}(z) \,/\, Q_{mn}(z)$, where

$$P_{mn}(z) = U_{mn}(z) \,/\, D(z)$$

$$Q_{mn}(z) = V_{mn}(z) \,/\, D(z) \,,$$

and in addition $Q_{mn}(0) = 1$.

That is,

$$P_{mn}(z) = z^{-\lambda}\, S_{mn}(z) = U^{*}_{m-\lambda,n-\lambda}(z)$$

$$Q_{mn}(z) = z^{-\lambda}\, T_{mn}(z) = V^{\bullet}_{m-\lambda,n-\lambda}(z) \; ,$$

where $U^{\bullet}_{m-\lambda,n-\lambda}(z)$ and $V^{\bullet}_{m-\lambda,n-\lambda}(z)$ are defined in (2.20). Thus, the Pade fraction of type (m,n) always exists and is unique. However, $P_{mn}(z) \,/\, Q_{mn}(z)$ is a member of the family of Pade forms of type (m,n) for $A(z)$ given by (2.18) and (2.19) only if $l = 2\lambda$, or equivalently, if $k \geq 2\lambda$ in equation (2.14). Thus, in the case that $k \geq 2\lambda$, the Pade fraction is obtained from the Pade form by setting $\alpha_0 = 1, \alpha_1 = \cdots = \alpha_{l-\lambda} = 0$ in equation (2.18) and (2.19). In the remaining case that $k < 2\lambda$, the Pade fraction $P_{mn}(z) \,/\, Q_{mn}(z)$ does not satisfy the systems (2.5) and (2.6).

Baker's [*BAK*75] perspective is different in a very subtle way. He requires that $P_{mn}(z) \,/\, Q_{mn}(z)$ satisfies (2.5) and (2.6) and in addition $Q_{mn}(0) = 1$ and $GCD(P_{mn}, Q_{mn}) = 1$. But once again, from Theorem 2.5, this is possible only if $k \geq 2\lambda$, in which case $P_{mn}(z) \,/\, Q_{mn}(z)$ is obtained by setting $\alpha_0 = 1, \alpha_1 = \cdots = \alpha_{l-\lambda} = 0$ in the equations (2.18) and (2.19). Thus, from Baker's point of view, a Pade fraction may not exist, but whenever it does, it is unique and satisfies systems (2.5) and (2.6).

CHAPTER 3

COMPUTATION OF OFF-DIAGONAL SCALED PADE FRACTIONS

3.1. Introduction

In this chapter, derived is a new relationship between three scaled Pade fractions lying along an off-diagonal path of the Pade table for a power series $A(z)$. This relationship is described in Section 3.2 and is used in Section 3.3 to develop an algorithm to iteratively compute a sequence of scaled Pade fractions along the off-diagonal path. However, the algorithm is not totally iterative, since the relationship recursively involves a scaled Pade fraction of a different power series $\bar{A}(z)$, computed from $A(z)$. By doubling the step-size along the off-diagonal path at each iteration, it is shown that the complexity of the algorithm for computing the n-th scaled Pade fraction along the path is $(n \log^2 n)$, assuming fast polynomial methods are used.

The algorithm of Section 3.3 can be used to compute scaled Pade fractions for the quotient $-A(z)/B(z)$ of two power series $A(z)$ and $B(z)$. This can be accomplished by formally computing the inverse of $B(z)$, multiplying it by $-A(z)$, and then applying the algorithm to the result. The dominating cost of this procedure is the cost of computing the inverse of $B(z)$. This cost, however, can be eliminated by an obvious modification of the algorithm, which is given in Section 3.4.

If the step-size along the off-diagonal path is shortened, all scaled Pade fractions along the path can be computed. The recursive call alluded to earlier then becomes trivial, and the resulting algorithm, given in Section 3.5, becomes truly iterative. In the case that the path is along the diagonal, the algorithm is identical to the one given by Cabay and Kao [CAB83]. This algorithm cannot take advantage of fast polynomial methods, and has complexity $O(n^2)$.

3.2. Preliminary Results

The scaled Pade fractions can be arranged in a doubly infinite array as follows:

Definition: The collection of all scaled Pade fractions of type (m,n) for $A(z) \in \mathbf{U}$, given by

$$\Gamma(A) = \left| \begin{array}{ccccc} \gamma_{-1,-1} & \gamma_{-1,0} & \gamma_{-1,1} & \cdots & \gamma_{-1,n} \cdots \\ \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,n} \cdots \\ \gamma_{1,-1} & \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,n} \cdots \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \gamma_{m,-1} & \gamma_{m,0} & \gamma_{m,1} & \cdots & \gamma_{m,n} \cdots \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{array} \right. \tag{3.1}$$

is called the (extended) **scaled Pade table**[1] for $A(z)$. The modifier extended denotes the inclusion of the first row and column in the table which are defined as follows:

$S_{m,-1}(z) = -z^m$ and $T_{m,-1}(z) = 0$, for $n \geq -1$, and

$S_{-1,n}(z) = 0$ and $T_{-1,n}(z) = -z^n$, for $n \geq 0$. ■

For computational purposes, define the N-truncated, scaled Pade table for $A(z)$ to be

$$\Gamma_N(A) = \left| \begin{array}{ccccc} \gamma_{-1,-1} & \gamma_{-1,0} & \gamma_{-1,1} & \cdots & \gamma_{-1,N} \\ \gamma_{0,-1} & \gamma_{0,0} & \gamma_{0,1} & \cdots & \gamma_{0,N} \\ \gamma_{1,-1} & \gamma_{1,0} & \gamma_{1,1} & \cdots & \gamma_{1,N} \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \end{array} \right| . \tag{3.2}$$

Let m and n be non-negative integers such that $n > N$. The next few results are concerned

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[1]Note that the (extended) scaled Pade table corresponds to the usual definition of the Pade table given, for example, in Gragg [*GRA*72] with the exception that the scaling factor z^λ in (2.21) and (2.22) does not appear.

with the construction of the scaled Pade fraction $\gamma_{mn}(z)$, given that $\Gamma_N(A)$ already exists. Without loss of generality, assume that $m \geq n$ (otherwise, the same arguments can be applied to $1/A(z) \in \mathbf{U}$). Let

$$M = N + (m - n) . \tag{3.3}$$

Then, (m, n) and (M, N) both lie along the $(m-n)$th off-diagonal path of the scaled Pade table $\Gamma(A)$ (see Figure 3.1).

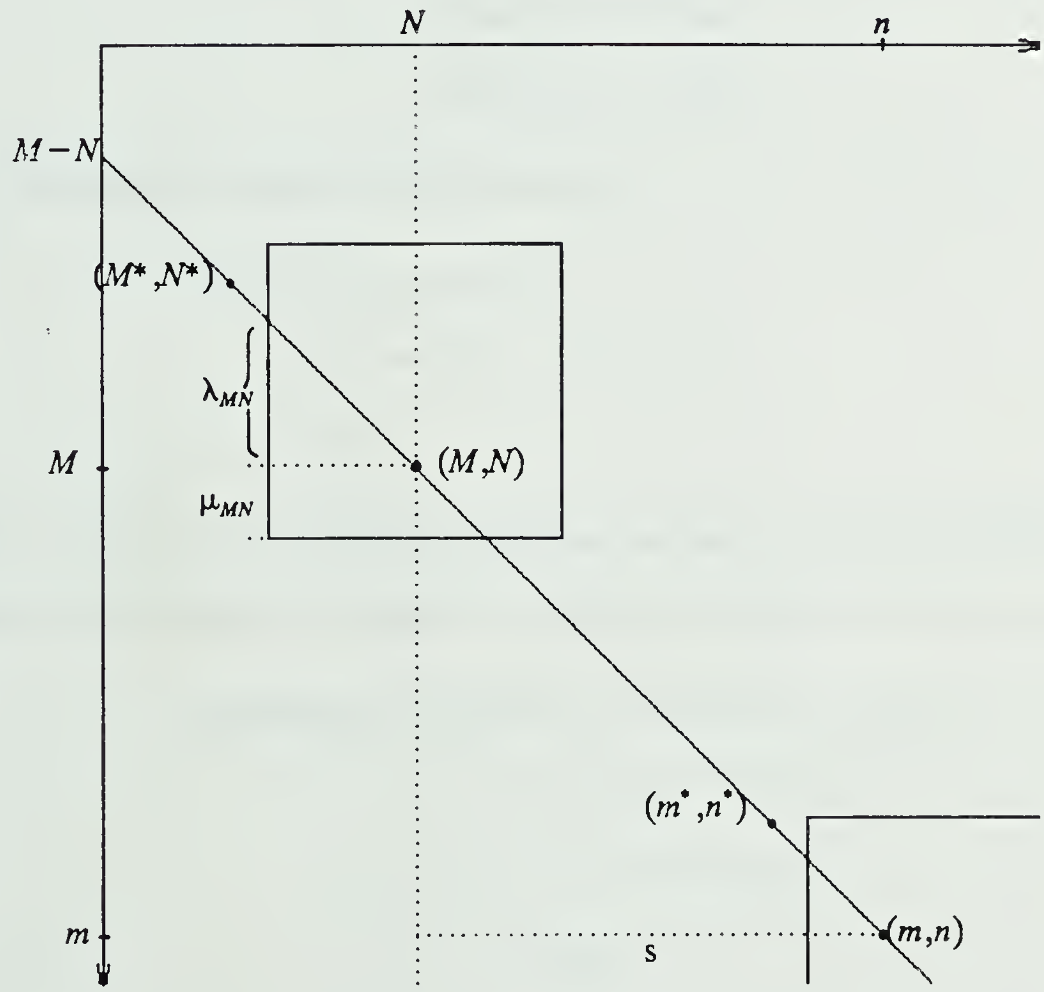

Figure 3.1

For the scaled Pade fraction of type (M,N) for $A(z)$, equation (2.3) becomes

$$A(z) \cdot T_{MN}(z) - S_{MN}(z) = z^{M+N+\mu_{MN}+1} R_1(z) , \tag{3.4}$$

where $\mu_{MN} \geq 0$, and $R_1(0) \neq 0$ if $\mu_{MN} < \infty$. Let

$$z^{\lambda_{MN}} = GCD(S_{MN}, T_{MN}) \,. \tag{3.5}$$

To construct $\gamma_{mn}(z)$, two separate cases, $\mu_{MN} \geq (n-N)$ and $\mu_{MN} < (n-N)$, arise. These two cases are considered seperately in Theorem 3.1 and Theorem 3.4.

Theorem 3.1: If $\mu_{MN} \geq (n-N)$ in (3.4), then the scaled Pade fraction $\gamma_{mn}(z) = S_{mn}(z)/T_{mn}(z)$ is given by

$$S_{mn}(z) = z^{n-N}\, S_{MN}(z) \tag{3.6}$$

$$T_{mn}(z) = z^{n-N}\, T_{MN}(z) \,. \tag{3.7}$$

Proof: Clearly from relation (3.3), it follows that

$$\begin{aligned} \partial(S_{mn}) &= n-N+\partial(S_{MN}) \\ &\leq n+M-N \\ &= m. \end{aligned}$$

Similarly, $\partial(T_{mn}) \leq n$. Furthermore, it is clear that $\min\{\, m-\partial(S_{mn}),\, n-\partial(T_{mn}) \,\} = 0$ from the equality, $\min\{\, M-\partial(S_{MN}), M-\partial(T_{MN}) \,\} = 0$. Using the fact that $(n-N) \leq \mu_{MN}$, it follows that

$$\begin{aligned} A(z)T_{mn}(z) - S_{mn}(z) &= z^{n-N}\,(A(z)T_{MN}(z) - S_{MN}(z)) \\ &= z^{n-N}\,(z^{M+N+\mu_{MN}+1}R_1(z)) \\ &= z^{m+N+\mu_{MN}+1}R_1(z) \\ &= O(z^{m+n+1}) \,. \end{aligned}$$

Finally, $GCD(S_{mn}, T_{mn}) = z^{n-N}\, GCD(S_{MN}, T_{MN}) = z^{n-N+\lambda_{MN}}$.

Consequently, $\gamma_{mn}(z) = S_{mn}(z) \,/\, T_{mn}(z)$ given by (3.6) and (3.7) is the scaled Pade fraction of type (m,n) for $A(z)$. ■

For the case $\mu_{MN} < (n-N)$, let

$$M^* = M - \lambda_{MN} - 1 , \tag{3.8}$$

$$N^* = N - \lambda_{MN} - 1 \tag{3.9}$$

(see Figure 3.1). Clearly, the scaled fraction $\gamma_{M^*N^*}(z)$ for $A(z)$ satisfies

$$A(z) \cdot T_{M^*N^*}(z) - S_{M^*N^*}(z) = z^{M^*+N^*+1} R_0(z) , \tag{3.10}$$

where $R_0(0) \neq 0$. In addition, by Theorem 2.7 and by the definition of M^* and N^*, it follows that

$$S_{MN}(z) \,/\, T_{MN}(z) \neq S_{M^*N^*}(z) \,/\, T_{M^*N^*}(z) . \tag{3.11}$$

From the two unit power series, $R_1(z)$ and $R_0(z)$ given by (3.4) and (3.10), respectively, construct the unique power series

$$\bar{A}(z) = R_0(z) \,/\, R_1(z) . \tag{3.12}$$

Associated with $\bar{A}(z) \epsilon \mathbf{U}$, let

$$\bar{m} = n - N + \lambda_{MN} , \tag{3.13}$$

$$\bar{n} = n - N - \mu_{MN} - 1 . \tag{3.14}$$

Now assume that the $\bar{n}$-truncated scaled Pade table, $\Gamma_{\bar{n}}(\bar{A})$, for $\bar{A}(z)$ has been constructed, as well. That is, assume that the $(\bar{m},\bar{n})$ scaled Pade fraction, $\bar{\gamma}_{\bar{m}\bar{n}}(z) = \bar{S}_{\bar{m}\bar{n}}(z) \,/\, \bar{T}_{\bar{m}\bar{n}}(z)$, such that

$$\bar{A}(z) \cdot \bar{T}_{\bar{m}\bar{n}}(z) - \bar{S}_{\bar{m}\bar{n}}(z) = O(z^{\bar{m}+\bar{n}+1}) \tag{3.15}$$

is available.

The scaled Pade fractions $\gamma_{MN}(z)$, $\gamma_{M^*N^*}(z)$ and $\bar{\gamma}_{\bar{m}\bar{n}}(z)$ provide sufficient information to obtain directly a Pade form of type (m,n) for $A(z)$. This Pade form is constructed in Lemma 3.2 below. It is shown, later in Theorem 3.4, that this form is also the scaled Pade fraction of type (m,n) for $A(z)$.

Lemma 3.2: Let $\mu_{MN} < (n-N)$ in equation (3.4), and let

$$S_{mn}(z) = z^{-\lambda_{MN}} S_{MN}(z)\, \bar{S}_{\bar{m}\,\bar{n}}(z) - z^{\alpha}\, S_{M^*N^*}(z)\, \bar{T}_{\bar{m}\,\bar{n}}(z) \tag{3.16}$$

$$T_{mn}(z) = z^{-\lambda_{MN}} T_{MN}(z)\, \bar{S}_{\bar{m}\,\bar{n}}(z) - z^{\alpha}\, T_{M^*N^*}(z)\, \bar{T}_{\bar{m}\,\bar{n}}(z), \tag{3.17}$$

where $\alpha = \lambda_{MN} + \mu_{MN} + 2$. Then $S_{mn}(z) \,/\, T_{mn}(z)$ is a Pade form of type (m,n) for $A(z)$.

Proof:

$$\begin{aligned}
\partial(S_{mn}) &= \max\{\partial(z^{-\lambda_{MN}} S_{MN} \bar{S}_{\bar{m}\bar{n}}),\ \partial(z^{\alpha} S_{M^*N^*} \bar{T}_{\bar{m}\bar{n}})\} \\
&\le \max\{ -\lambda_{MN} + M + \bar{m},\ \alpha + M^* + \bar{n}\} \\
&= \max\{ -\lambda_{MN} + M + (n - N + \lambda_{MN}), \\
&\qquad\qquad (\lambda_{MN} + \mu_{MN} + 2) + (M - \lambda_{MN} - 1) + (n - N - \mu_{MN} - 1)\} \\
&= \max\{m,\ m\} \\
&= m.
\end{aligned}$$

Similarly, $\partial(T_{mn}) \le n$. Moreover, the fact that $S_{MN}(z) \,/\, T_{MN}(z)$, $S_{M^*N^*}(z) \,/\, T_{M^*N^*}(z)$ and $\bar{S}_{\bar{m}\bar{n}}(z) \,/\, \bar{T}_{\bar{m}\bar{n}}(z)$ are the scaled Pade fractions yields immediately that

$$\min\{m - \partial(S_{mn})\ ,\ n - \partial(T_{mn})\} = 0.$$

Furthermore, from (3.4), (3.10), (3.12) and (3.15), it follows that

$$\begin{aligned}
A(z) \cdot T_{mn}(z) - S_{mn}(z) &= A(z)\, \{z^{-\lambda_{MN}} T_{MN}(z) \bar{S}_{\bar{m}\bar{n}}(z) - z^{\alpha}\, T_{M^*N^*}(z) \bar{T}_{\bar{m}\bar{n}}(z)\} \\
&\qquad - \{z^{-\lambda_{MN}} S_{MN}(z) \bar{S}_{\bar{m}\bar{n}}(z) - z^{\alpha}\, S_{M^*N^*}(z) \bar{T}_{\bar{m}\bar{n}}(z)\} \\
&= z^{-\lambda_{MN}} \bar{S}_{\bar{m}\bar{n}}(z)\, \{A(z) T_{MN}(z) - S_{MN}(z)\} \\
&\qquad - z^{\alpha}\, \bar{T}_{\bar{m}\bar{n}}(z) \{A(z) T_{M^*N^*}(z) - S_{M^*N^*}(z)\} \\
&= z^{-\lambda_{MN}} \bar{S}_{\bar{m}\bar{n}}(z) \{R_1(z)\, z^{M+N+\mu_{MN}+1}\} - z^{\alpha}\, \bar{T}_{\bar{m}\bar{n}}(z) \{R_0(z)\, z^{M^*+N^*+1}\}
\end{aligned}$$

$$= z^{M+N+\mu_{MN}-\lambda_{MN}+1}\{R_1(z)\,\bar{S}_{\bar{m}\bar{n}}(z) - R_0(z)\,\bar{T}_{\bar{m}\bar{n}}(z)\}$$

$$= -R_1(z)\,z^{M+N+\mu_{MN}-\lambda_{MN}+1}\,\{\,\bar{T}_{\bar{m}\bar{n}}(z)\,R_0(z)\,/\,R_1(z) - \bar{S}_{\bar{m}\bar{n}}(z)\,\}$$

$$= -R_1(z)\,z^{M+N+\mu_{MN}-\lambda_{MN}+1}\,O(z^{\bar{m}+\bar{n}+1})$$

$$= O(z^{M+N+\mu_{MN}-\lambda_{MN}+1+\bar{m}+\bar{n}+1})$$

$$= O(z^{m+n+1}). \qquad \blacksquare$$

In order that $S_{mn} \,/\, T_{mn}$ in (3.16) and (3.17) be a scaled Pade fraction of type (m, n), it remains to show that $GCD(S_{mn}, T_{mn}) = z^{\lambda_{mn}}$ for some λ_{mn}. With this intent, consider again the $(\bar{m}, \bar{n})-th$ entry of $\Gamma_{\bar{n}}(\bar{A})$, and let

$$\bar{m}^{*} = \bar{m} - \bar{\lambda}_{\bar{m}\bar{n}} - 1 \tag{3.18}$$

$$\bar{n}^{*} = \bar{m} - \bar{\lambda}_{\bar{m}\bar{n}} - 1, \tag{3.19}$$

where

$$z^{\bar{\lambda}_{\bar{m}\bar{n}}} = GCD(\bar{S}_{\bar{m}\bar{n}}, \bar{T}_{\bar{m}\bar{n}}). \tag{3.20}$$

The scaled Pade fraction $\bar{\gamma}_{\bar{m}^{*},\,\bar{n}^{*}}(z)$ for $\bar{A}(z)$ satisfies

$$\begin{aligned} \bar{A}\bar{T}_{\bar{m}^{*}\bar{n}^{*}} - \bar{S}_{\bar{m}^{*}\bar{n}^{*}} &= O_E(z^{\bar{m}^{*}+\bar{n}^{*}+1}) \\ &= O_E(z^{\bar{m}+\bar{n}-2\bar{\lambda}_{\bar{m}\bar{n}}-1}),^{2} \end{aligned} \tag{3.21}$$

and, in addition,

$$\frac{\bar{S}_{\bar{m}\bar{n}}(z)}{\bar{T}_{\bar{m}\bar{n}}(z)} \neq \frac{\bar{S}_{\bar{m}^{*}\bar{n}^{*}}(z)}{\bar{T}_{\bar{m}^{*}\bar{n}^{*}}(z)}\,. \tag{3.22}$$

---

[2] $R(z) = O_E(\,z^{k}\,)$ means that $R(z)$ is a power series whose first non-zero coefficient is the coefficient of z^{k}, exactly.

Lemma 3.3: Let

$$S_{m^*n^*}(z) = z^{-\lambda_{MN}} S_{MN}(z)\, \bar{S}_{\bar{m}^*\bar{n}^*}(z) - z^{\alpha} S_{M^*N^*}(z)\, \bar{T}_{\bar{m}^*\bar{n}^*}(z) \tag{3.23}$$

$$T_{m^*n^*}(z) = z^{-\lambda_{MN}} T_{MN}(z)\, \bar{S}_{\bar{m}^*\bar{n}^*}(z) - z^{\alpha} T_{M^*N^*}(z)\, \bar{T}_{\bar{m}^*\bar{n}^*}(z) \tag{3.24}$$

where $\alpha = \lambda_{MN} + \mu_{MN} + 2$. Then $\gamma_{m^*n^*}(z) = S_{m^*n^*}(z) / T_{m^*n^*}(z)$ is a Pade form of type (m^*, n^*) for $A(z)$, where

$$m^* = m - \bar{\lambda}_{\bar{m}\bar{n}} - 1 \tag{3.25}$$

$$n^* = n - \bar{\lambda}_{\bar{m}\bar{n}} - 1. \tag{3.26}$$

Proof: The proof is identical to the proof of Lemma 3.2. ■

Theorem 3.4: Let $\mu_{MN} < (n - N)$ in equation (3.4), and let

$$\gamma_{mn}(z) = S_{mn}(z) / T_{mn}(z)$$

be defined by (3.16) and (3.17). Then $\gamma_{mn}(z)$ is a scaled Pade fraction of type (m, n) for $A(z)$.

Proof: Let

$$G_{mn}(z) = GCD(S_{mn}, T_{mn}).$$

We first show that $\partial(G_{mn}) \leq \bar{\lambda}_{\bar{m}\bar{n}}$, where $\bar{\lambda}_{\bar{m}\bar{n}}$ is given by (3.20). Suppose that $\partial(G_{mn}) > \bar{\lambda}_{\bar{m}\bar{n}}$, and proceed by contradiction. Let

$$U_{m^*n^*}(z) = z^{\partial(G_{mn}) - \bar{\lambda}_{\bar{m}\bar{n}} - 1} S_{mn}(z) / G_{mn}(z)\,,$$

$$V_{m^*n^*}(z) = z^{\partial(G_{mn}) - \bar{\lambda}_{\bar{m}\bar{n}} - 1} T_{mn}(z) / G_{mn}(z),$$

where m^* and n^* are given by (3.25) and (3.26). Then $\partial(U_{m^*n^*}) \leq m^*$, $\partial(V_{m^*n^*}) \leq n^*$, and

$$A(z) \cdot V_{m^*n^*}(z) - U_{m^*n^*}(z)$$

$$= z^{\partial(G_{mn}) - \bar{\lambda}_{\bar{m}\bar{n}} - 1}\{A(z)T_{mn}(z) - S_{mn}(z)\} / G_{mn}(z)$$

$$= O(z^{(-\bar{\lambda}_{\bar{m}\bar{n}} - 1) + (m + n + 1)})$$

$$= O(z^{m^* + n^* + \bar{\lambda}_{\bar{m}\bar{n}} + 1}).$$

Thus, $U_{m^* n^*}(z) / V_{m^* n^*}(z)$ is a Pade form of type (m^*, n^*) for $A(z)$.

But $S_{m^* n^*}(z) / T_{m^* n^*}(z)$, given by Lemma 3.3, is also a Pade form of type (m^*, n^*). Then, from the general structure of Pade forms given in equations (2.18) and (2.19) of Corollary 2.6, it follows that

$$S_{m^* n^*}(z) / T_{m^* n^*}(z) = U_{m^* n^*}(z) / V_{m^* n^*}(z)$$
$$= S_{mn}(z) / T_{mn}(z),$$

or, equivalently, that

$$S_{m^* n^*}(z)\, T_{mn}(z) - S_{mn}(z)\, T_{m^* n^*}(z) = 0. \qquad (3.27)$$

Replacing $S_{mn}(z) / T_{mn}(z)$ and $S_{m^* n^*} / T_{m^* n^*}$ in equation (3.27) by the expanded forms (3.16), (3.17), (3.23) and (3.24), it follows that

$$z^{\alpha} z^{-\lambda_{MN}}\{S_{MN}(z)\, T_{M^* N^*}(z) - S_{M^* N^*}(z)\, T_{MN}(z)\} \cdot \{\bar{S}_{\bar{m}\bar{n}}(z)\, \bar{T}_{\bar{m}^* \bar{n}^*}(z) - \bar{S}_{m^* n^*}(z)\, \bar{T}_{\bar{m}\bar{n}}(z)\} = 0\,.$$

Thus, either $S_{MN}(z) / T_{MN}(z) = S_{M^* N^*}(z) / T_{M^* N^*}(z)$, or $\bar{S}_{\bar{m}\bar{n}}(z) / \bar{T}_{\bar{m}\bar{n}}(z) = \bar{S}_{\bar{m}^* \bar{n}^*}(z) / \bar{T}_{\bar{m}^* \bar{n}^*}(z)$, which contradicts (3.11) and (3.22).

Thus, $\partial(G_{mn}) \leq \bar{\lambda}_{\bar{m}\bar{n}}$. But,

$$z^{\bar{\lambda}_{\bar{m}\bar{n}}} = GCD(\bar{S}_{\bar{m}\bar{n}}, \bar{T}_{\bar{m}\bar{n}}),$$

which implies that $z^{\bar{\lambda}_{\bar{m}\bar{n}}}$ divides both $S_{mn}(z)$ and $T_{mn}(z)$ in equations (3.16) and (3.17). That is, $z^{\bar{\lambda}_{\bar{m}\bar{n}}}$ divides $G_{mn}(z)$, and consequently

$$G_{mn}(z) = z^{\bar{\lambda}_{\bar{m}\bar{n}}}.$$

Thus, $S_{mn}(z) / T_{mn}(z)$ given in Lemma 3.2 is not only a Pade form of type (m,n) for $A(z)$, but also the scaled Pade fraction of type (m,n) for $A(z)$, where

$$GCD(S_{mn}\ , T_{mn}) = z^{\lambda_{mn}}$$

and $\lambda_{mn} = \bar{\lambda}_{\bar{m}\bar{n}}$. ■

Theorem 3.5: Let $\mu_{MN} < (n-N)$ in (3.4) and let $\gamma_{mn}(z) = S_{mn}(z) / T_{mn}(z)$ be the scaled Pade fraction of type (m,n) for $A(z)$. If

$$m^* = m - \lambda_{mn} - 1 \tag{3.28}$$

and

$$n^* = n - \lambda_{mn} - 1\ , \tag{3.29}$$

where

$$z^{\lambda_{mn}} = GCD(S_{mn}\ , T_{mn})\ , \tag{3.30}$$

then the scaled Pade fraction of type (m^*,n^*) for $A(z)$ is $\gamma_{m^*n^*}(z) = S_{m^*n^*}(z) / T_{m^*n^*}(z)$, where $S_{m^*n^*}(z)$ and $T_{m^*n^*}(z)$ are given by equation (3.23) and (3.24).

Proof: The theorem follows using arguments identical to those of proof of Theorem 3.4 , and using the results of Lemma 3.3. ■

A simple example for the off-diagonal computation is presented.

Example: Let $A(z) = 1 + z^4 + z^5 + z^9 + z^{10} + 2z^{15} + \cdots$. This example constructs the scaled Pade fraction $\gamma_{7,6}(z) = S_{7,6}(z) / T_{7,6}(z)$ of type (7,6) for $A(z)$. Since $m - n = 1$, the construction proceeds along the 1st off-diagonal path of the scaled Pade table $\Gamma(A)$.

Assume that $\Gamma_3(A)$ is already available, from which it can be determined that the scaled Pade fraction $\gamma_{4,3}(z)$ of type $(M,N) = (4,3)$ is given by

$$\gamma_{4,3}(z) = \frac{S_{4,3}(z)}{T_{4,3}(z)} = \frac{1 - z + z^2 - z^3 + z^4}{1 - z + z^2 - z^3} . \tag{3.31}$$

From (3.4), the residual for $S_{4,3}(z) \,/\, T_{4,3}(z)$ is given by

$$A(z)\, T_{4,3}(z) - S_{4,3}(z) = R_1(z)\, z^{4+3+1} , \tag{3.32}$$

where $R_1(z) = -1 + z - z^5 + 2z^7 + \cdots$.

Consequently, equation (3.32) yields that $\mu_{4,3} = 0$. Since $\mu_{4,3} < n - N$, Theorem 3.4 is applicable. Observe that

$$z^{\lambda_{MN}} = z^0 = GCD(S_{4,3}\, , T_{4,3}),$$

and consequently the predecessor of $\gamma_{4,3}(z)$ along the 1st off-diagonal path is $\gamma_{M^*N^*}(z) = \gamma_{3,2}(z)$. Therefore, $\gamma_{3,2}(z)$ is contained in $\Gamma_3(A)$, and is found to be

$$\gamma_{3,2}(z) = \frac{S_{3,2}(z)}{T_{3,2}(z)} = \frac{z^2}{z^2} .$$

The residual for $S_{3,2}(z) \,/\, T_{3,2}(z)$ is given from

$$A(z)\, T_{3,2}(z) - S_{3,2}(z) = R_0(z)\, z^{3+2+1} , \tag{3.33}$$

where $R_0(z) = 1 + z + z^5 + z^6 + \cdots$.

The two residuals $R_1(z)$ and $R_0(z)$ give the residual power series $\bar{A}(z) \in \mathbf{U}$, where

$$\begin{aligned} \bar{A}(z) &= R_0(z) \,/\, R_1(z) \\ &= -1 - 2z - 2z^2 - 2z^3 - \cdots . \end{aligned} \tag{3.34}$$

Using equation (3.13) and (3.14),

$$\bar{m} = n - N + \lambda_{MN} = 3$$

$$\bar{n} = n - N - \mu_{MN} - 1 = 2,$$

and in order to apply Theorem 3.4, it is therefore required to obtain the scaled Pade fraction $\bar{\gamma}_{3,2}(z)$ of type (3,2) and its predecessor for $\bar{A}(z)$. Assuming that $\Gamma_3(\bar{A})$ is available, from it can be

determined that

$$\bar{\gamma}_{3,2}(z) = \frac{\bar{S}_{3,2}(z)}{\bar{T}_{3,2}(z)} = \frac{-z - z^2}{z - z^2} . \tag{3.35}$$

Similarly, the predecessor of $\bar{\gamma}_{3,2}(z)$ along the first off-diagonal path of $\Gamma_3(\bar{A})$ is given by $\bar{\gamma}_{1,0}(z) = \bar{S}_{1,0}(z) \,/\, \bar{T}_{1,0}(z) = (-1 - 2z)/1$.

Now by applying the formulae (3.16) and (3.17), the (7,6) entry of $\Gamma(A)$ is computed by

$$S_{7,6}(z) = (1 - z + z^2 - z^3 + z^4)\cdot(-z - z^2) - z^\alpha\,(z^2)\cdot(z - z^2) ,$$

$$T_{7,6}(z) = (1 - z + z^2 - z^3)\cdot(-z - z^2) - z^\alpha\,(z^2)\cdot(z - z^2) ,$$

where $\alpha = \lambda_{4,3} + \mu_{4,3} + 2 = 2$. Thus,

$$\gamma_{7,6}(z) = S_{7,6}(z) \,/\, T_{7,6}(z) = z(-1 - z^4) \,/\, z(-1 + z^5). \tag{3.36}$$

Note that $\bar{\gamma}_{1,0}(z)$ is not used for computing $\gamma_{7,6}(z)$. It is used instead in formulae (3.23) and (3.24) for computing the predecessor of $\gamma_{7,6}(z)$. Since

$$z^{\lambda_{mn}} = z^{\bar{\lambda}_{\bar{m}\bar{n}}} = z^1 = GCD(\bar{S}_{3,2} \, , \, \bar{T}_{3,2}),$$

it is known that the predecessor is $\gamma_{5,4}(z)$, which is determined by (3.23) and (3.24) to be

$$\gamma_{5,4}(z) = \frac{S_{5,4}(z)}{T_{5,4}(z)} = \frac{(-1 - z + z^2 - z^3 - 2z^5)}{(-1 - z + z^2 - z^3 + z^4)} . \qquad \blacksquare$$

3.3. Fast Off-diagonal Algorithm for a Single Power Series

3.3.1. The Algorithm

The algorithm given in this section constructs the scaled Pade fraction γ_{mn} of type (m,n) for $A(z)$ in a quadratic fashion. The iteration assumes the existence of

$$\gamma_{MN}(z) = S_{MN}(z) \,/\, T_{MN}(z) , \tag{3.37}$$

where $M = N+(m-n)$, and where for notational convenience we set

$$S_1 = S_{MN}(z) \tag{3.38}$$

and

$$T_1 = T_{MN}(z) \ . \tag{3.39}$$

Also assumed to exist is the predecessor

$$\gamma_{M^*N^*}(z) = S_{M^*N^*}(z) \,/\, T_{M^*N^*}(z) \tag{3.40}$$

of $\gamma_{MN}(z)$ on the the $(m-n)$-th off-diagonal path of $\Gamma(A)$, where

$$M^* = M - \lambda_{MN} - 1 \ ,$$

$$N^* = N - \lambda_{MN} - 1$$

and

$$z^{\lambda_{MN}} = GCD(S_{MN}, T_{MN}) \ .$$

Again for notational convenience, we set

$$S_0 = S_{M^*N^*}(z) \tag{3.41}$$

and

$$T_0 = T_{M^*N^*}(z) \ . \tag{3.42}$$

To advance the solution from N to $N+s$ (that is, to construct $\gamma_{M+s,N+s}(z)$), where s is the step size, the algorithm first computes μ_{MN} such that

$$A(z)T_1 - S_1 \bmod z^{M+N+2s+\lambda_{MN}+1} = z^{M+N+\mu_{MN}+1} R_1(z) \ ,$$

where

$$z^{\lambda_{MN}} = GCD(\ S_1\ ,\ T_1\ )$$

and $R_1(0) \neq 0$ if $\mu_{MN} < 2s + \lambda_{MN}$.

If $\mu_{MN} \geq s$, then $\gamma_{M+s,N+s}(z)$ is constructed trivially by means of Theorem 3.1. Otherwise,

Theorems 3.4 and 3.5 are applied.

ALGORITHM 1: **OFFDIAG**

INPUT: A, m, n , where

(1) m and n are non-negative integers with $m \geq n$, and

(2) A is a unit power series . (Note that only $A \; mod \; z^{m+n+1}$ is required).

OUTPUT: $\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix}$, where

(1) S_1 / T_1 is the scaled Pade fraction of type (m, n) for A, and

(2) S_0 / T_0 is the scaled Pade fraction of type $(m-\lambda_{mn}-1, n-\lambda_{mn}-1)$ for A, given that

$$z^{\lambda_{mn}} = GCD(S_1, T_1) \; .$$

Step 1: # Initialization #

$$\begin{aligned} i &\leftarrow -1 \\ M &\leftarrow (m-n) \\ N &\leftarrow 0 \end{aligned}$$

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} A(z) \; mod \; z^{M+1} & -z^{M-1} \\ 1 & 0 \end{bmatrix}$$

Step 2: # Calculation of step-size #

$$\begin{aligned} i &\leftarrow i+1 \\ s &\leftarrow \min \{2^i - N \, , \, n-N\} \end{aligned}$$

Step 3: # Termination criterion #

If $s = 0$ then exit

Step 4: # Calculation of scaling factor for S_1 / T_1 #

Determine λ_{MN} such that $z^{\lambda_{MN}} = GCD(S_1, T_1)$

Step 5: # Computation of residual for $\gamma_{MN} = S_1 / T_1$ #

Compute μ_{MN} and R_1 such that

$$(A\,T_1 - S_1) \; mod \; z^{M+N+2s+\lambda_{MN}+1} = z^{M+N+\mu_{MN}+1} R_1 \, ,$$

where $R_1(0) \neq 0$ if $\mu_{MN} < 2s + \lambda_{MN}$.

Step 6: # Identification of Cases #

if $\mu_{MN} \geq s$

then # Case of Theorem 3.1 #

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \begin{bmatrix} z^s & 0 \\ 0 & 1 \end{bmatrix}$$

go to step 12

else # Case of Theorem 3.4 and 3.5 #

go to step 7

Step 7: # Calculation of degrees for residual scaled Pade fractions #

$$\bar{m} \leftarrow s + \lambda_{MN}$$
$$\bar{n} \leftarrow s - \mu_{MN} - 1$$

Step 8: # Computation of residual for $\gamma_{M^*N^*}(z) = S_0 / T_0$ #

Compute R_0 such that

$$(A(z)\cdot T_0 - S_0) \ mod \ z^{M+N+\bar{m}+\bar{n}-2\lambda_{MN}} = z^{M+N-2\lambda_{MN}-1} R_0 ,$$

where $R_0(0) \neq 0$.

Step 9: # Computation of residual power series #

$$\bar{A}(z) \leftarrow R_0(z) / R_1(z) \ \ (mod \ z^{\bar{m}+\bar{n}+1})$$

Step 10: # Computation of residual scaled Pade fractions #

$$\begin{bmatrix} \bar{S}_1 & \bar{S}_0 \\ \bar{T}_1 & \bar{T}_0 \end{bmatrix} \leftarrow \textbf{OFFDIAG} \ \ (\bar{A}(z), \bar{m}, \bar{n})$$

Step 11: # Advancement of scaled Pade fraction computation #

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \begin{bmatrix} z^{-\lambda_{MN}} & 0 \\ 0 & -z^{\lambda_{MN}+\mu_{MN}+2} \end{bmatrix} \begin{bmatrix} \bar{S}_1 & \bar{S}_0 \\ \bar{T}_1 & \bar{T}_0 \end{bmatrix}$$

Step 12: # Calculation of degrees of S_1 / T_1 #

$N \leftarrow N + s$
$M \leftarrow M + s$
go to step 2 ∎

3.3.2. Proof of algorithm validity

Let $\gamma_{mn}^{(A)}(z)$ be the scaled Pade fraction of type (m,n) for $A(z)$, where $m \geq n \geq -1$. As in (3.2), define

$$\Gamma_N(A) = \{ \ \gamma_{mn}^{(A)}(z) \ \mid m \geq n \ , N \geq n \geq -1 \ \} \tag{3.43}$$

to be the N-truncated Pade table for $A(z)$, with the additional provision that $m \geq n$.

Also, define

$$\Omega_N = \{\Gamma_N(A) \text{ for all } A\epsilon \mathbf{U}\} \tag{3.44}$$

for $N = 0,1, \cdots$.

The proof proceeds by induction on Ω_{2^i} for $i = 0,1,2, \cdots$. Clearly the algorithm is correct for Ω_0.

It is shown that if the algorithm is correct for Ω_N , then it is also correct for Ω_{N+L} ; where

$$L = \begin{cases} 1 & , \text{ if } N = 0 \\ N & , \text{ if } N = 2^i \quad , i \geq 0. \end{cases} \tag{3.45}$$

Let $A(z) \in \mathbf{U}$ be an arbitrary unit power series. It is shown that the algorithm correctly computes $\gamma_{mn}^{(A)}(z)$, where $m \geq n$ and $-1 \leq n \leq N+L$, given that it is correct for Ω_N .

If $-1 \leq n \leq N$, then $\gamma_{mn}^{(A)}(z) \epsilon\, \Gamma_N(A) \epsilon\, \Omega_N$, and by the inductive hypothesis the algorithm computes it correctly.

If $N < n \leq N+L$, let $M = N + (m-n)$. By the inductive hypothesis, the algorithm (using $\log_2 N$ iterations) correctly computes $\gamma_{MN}^{(A)}(z) = S_{MN}^{(A)}(z) \,/\, T_{MN}^{(A)}(z) \in \Gamma_N(A) \in \Omega_N$, and its predecessor $\gamma_{M^*N^*}^{(A)}(z)$. It remains to show that one more iteration correctly gives $\gamma_{mn}^{(A)}(z)$ and its predecessor $\gamma_{m^*n^*}^{(A)}(z)$.

Let $s = n - N$ be given by step 2 of the algorithm and let $z^{\lambda_{MN}} = GCD(S_{MN}^{(A)} , T_{MN}^{(A)})$ be given by step 4. Then step 5 yields $\mu_{MN} \geq 0$ and a polynomial $R_1(z)$ such that

$$(\, A(z)T_{MN}^{(A)}(z) - S_{MN}^{(A)}(z) \,) \bmod z^{M+N+2s+\lambda_{MN}+1} = R_1(z)\, z^{M+N+\mu_{MN}+1} , \tag{3.46}$$

where $\mu_{MN} \geq 0$, and $R_1(0) \neq 0$ if $\mu_{MN} < 2s + \lambda_{MN}$. Then two cases arise.

If $\mu_{MN} \geq s$, then by Theorem 3.1, step 5 correctly yields $\gamma_{mn}^{(A)}(z)$. Furthermore, from equations (3.6) and (3.7),

$$\begin{aligned} z^{\lambda_{mn}} &= GCD(S_{mn}^{(A)}(z)\ ,\ T_{mn}^{(A)}(z)) \\ &= z^{\lambda_{MN} + s} . \end{aligned}$$

Consequently, the predecessor of $\gamma_{mn}^{(A)}(z)$ is given by

$$\gamma_{m^\bullet n^\bullet}^{(A)}(z) = \gamma_{M^\bullet N^\bullet}^{(A)}(z) , \tag{3.47}$$

since

$$m^\bullet = m - \lambda_{mn} - 1 = M - \lambda_{MN} - 1 = M^\bullet$$

and

$$n^\bullet = n - \lambda_{mn} - 1 = N - \lambda_{MN} - 1 = N^\bullet .$$

If $\mu_{MN} < s$, then step 7 gives

$$\bar{n} = s - \mu_{MN} - 1 \leq N$$

and

$$\bar{m} = s + \lambda_{MN} \geq \bar{n} .$$

The polynomial $R_1(z)$, computed in step 5, therefore satisfies $R_1(0) \neq 0$ and is of degree $\bar{m} + \bar{n}$. In addition, step 8 produces a polynomial $R_0(z)$ of degree $\bar{m} + \bar{n}$ such that $R_0(0) \neq 0$. Consequently, enough terms are available in $R_1(z)$ and $R_0(z)$ to compute, in step 9, the residual unit power series $\bar{A}(z)$ *mod* $z^{\bar{m} + \bar{n} + 1}$ defined by (3.12). By the inductive hypothesis, step 10 therefore correctly computes $\gamma_{\bar{m}\,\bar{n}}^{(\bar{A})}(z) \in \Gamma_N(\bar{A}) \in \Omega_N$ and its predecessor. It now follows by Theorems 3.4 and 3.5 that step 11 correctly computes $\gamma_{mn}^{(A)}(z)$ and its predecessor. ∎

3.3.3. Cost Analysis

Let $C(m,n)$ be the cost of computing the scaled Pade fraction of type (m,n) and its predecessor for an arbitrary power series $A(z) \in \mathbf{U}$ using Algorithm 1. For the sake of simplicity, assume $0 \leq n \leq m \leq 2n$. The case that $m > 2n$ is considered later. In this section, asymptotic estimates of $C(m,n)$ are derived by counting the number of operations (additions, subtractions, multiplications and divisions in $\mathbf{F}$) performed by the algorithm. A detailed cost analysis and an implementation of the algorithm are described by Verheijen [VER83], who compares Algorithm 1 with other algorithms for calculating Pade fractions.

When obtaining the asymptotic cost estimates, it is assumed that the algorithm makes use of fast methods for polynomial and power series arithmetic. Using fast Fourier transforms, two polynomials with coefficients in the field $\mathbf{F}$ and of degree M and N, respectively, can be multiplicated in $O((M + N)\log(M + N))$ operations in $\mathbf{F}$. Using Newton's method and fast multiplication of polynomials, the first N terms of the quotient of two power series in $\mathbf{U}$ can be obtained in $O(N \log N)$ operations in $\mathbf{F}$. These and other fast methods for polynomial and power series arithmetic are described, for example, in Aho, Hopcraft and Ullman [AHO74] and in Lipson [LIP81].

Let $k = \lceil \log n \rceil$. Then it is easy to verify that the algorithm terminates after k iterations, and that after the execution of step 2 of iteration i

$$s = \begin{cases} 1 & , \ i = 0 \\ 2^{i-1} & , \ 0 < i < k \\ n - 2^{k-1} & , \ i = k. \end{cases} \tag{3.48}$$

Consequently, during iteration i,

$$N = \begin{cases} 0 & , \ i = 0 \\ 2^{i-1} & , \ \text{otherwise,} \end{cases} \tag{3.49}$$

and

$$M = N + (m-n)\ . \tag{3.50}$$

Assume that during iteration i, the scaled Pade fraction S_1 / T_1 of type (M,N) and its predecessor S_0 / T_0 are available.

The first nontrivial step requires the computation of the residual $R_1(z)$ in step 5. Let

$$A_1(z) = \sum_{j=0}^{M} a_j z^j \tag{3.51}$$

and

$$A_2(z) = \sum_{j=0}^{N+2s-1} a_{M+j+1} z^j\ . \tag{3.52}$$

With λ_{MN} given by step 4, it is known that

$$\begin{aligned} A(z)\, T_1 - S_1 \quad & mod \quad z^{M+N+2s+\lambda_{MN}+1} \\ &= A_1(z)\, T_1 + z^{M+1} A_2(z)\, T_1 - S_1 \quad mod \quad z^{M+N+2s+\lambda_{MN}+1} \\ &= O(z^{M+N+1}). \end{aligned} \tag{3.53}$$

Since $A_1 T_1$ and S_1 are both of degree at most $M + N$, then μ_{MN} and $R_1(z)$ can be obtained directly from $z^{M+1} A_2 T_1$. The product $A_2 T_1$ is a polynomial of at most degree $2N + 2s - 1 < 4N$. Thus, using fast polynomial multiplication, step 5 can be executed in $O(N \log N)$ operations.

If it is determined as a result of step 5 that $\mu_{MN} \geq s$, then step 6 is performed trivially and the iteration is complete. Otherwise, the algorithm continues in step 8 with the calculation of the residual R_0 of the predecessor scaled Pade fraction S_0 / T_0. Making the same observations as in step 4, it follows that R_0 can be obtained from the product of $A_3 T_0$, where

$$A_3 = \sum_{j=0}^{N+\bar{m}+\bar{n}-\lambda_{MN}} a_{M-\lambda_{MN}+j} z^j \tag{3.54}$$

and T_0 is a polynomial of degree at most $N - \lambda_{MN} - 1$. Since the degree of the product is

bounded by $2N + \bar{m} + \bar{n} - 2\lambda_{MN} - 1 < 4N$, step 8 can be executed in $O(N \log N)$ operations.

The computation of the residual power series $\bar{A}(z) \bmod z^{\bar{m}+\bar{n}+1}$ in step 9 requires the computation of the first $\bar{m} + \bar{n} = 2s + \lambda_{MN} - \mu_{MN} - 1 \le 3N$ terms of the quotient of R_0 / R_1. Again, this may be performed in $O(N \log N)$ operations.

In step 8, the recursive call of Algorithm 1 in order to compute the scaled Pade fraction of type $(\bar{m}, \bar{n})$ for $A(z)$, requires $C(\bar{m}, \bar{n})$ operations by assumption. For later purposes, it is important to observe that $0 \le \bar{n} \le N$ and that $\bar{n} \le \bar{m} \le 2N$.

The final non-trivial step requires eight polynomial multiplications to obtain the scaled Pade fraction of type $(M+s, N+s)$ and its predecessor for $A(z)$. Since $M \le 2N$, each of the polynomial products are of degree at most $M + s < 3N$. Consequently, step 8 an be executed in $O(N \log N)$ operations.

It is an easy matter to show that

$$C(m_1, n_1) \le C(m_2, n_2), \tag{3.55}$$

whenever $m_1 \le m_2$ and $n_1 \le n_2$. The total cost of the i-th iteration is then bounded by

$$\begin{aligned} & C(\bar{m}, \bar{n}) + c(2N) \log (2N) \\ \le\ & C(2N, N) + c(2N) \log (2N) \\ \le\ & C(2^i, 2^{i-1}) + ci2^i \end{aligned} \tag{3.56}$$

operations, for an appropriate constant c. Consequently, we have

Theorem 3.6. Given that $0 \le n \le m \le 2n$, Algorithm 1 can compute the scaled Pade fraction of type (m, n) for $A(z) \in \mathbf{U}$ in time $O(n \log^2 n)$.

Proof: Consider the recurrence relation

$$C(2^{k+1}, 2^k) = \sum_{i=1}^{k}[C(2^i, 2^{i-1}) + ci2^i], \qquad k \geq 1,$$

where c is a positive constant. Then

$$C(2^{k+1}, 2^k) = \sum_{i=1}^{k}[C(2^i, 2^{i-1}) + ci2^i] + C(2^k, 2^{k-1}) + ck2^k$$

$$= 2C(2^k, 2^{k-1}) + ck2^k.$$

With $n = 2^k$, results on recurrence relations (see Bentley, Hanken and Saxe [*BEN*80]) then yield

$$C(2n, n) = n[C(2, 1) + c\sum_{i=1}^{k} i]$$

$$= n[C(2, 1) + ck(k+1)/2]$$

$$= O(n \log^2 n).$$

The theorem now follows, since from equation (3.56) for m and n satisfying $0 \leq n \leq m \leq 2n$

$$C(m, n) \leq \sum_{i=1}^{k}[C(2^i, 2^{i-1}) + ci2^i]. \qquad \blacksquare$$

Lemma 3.7: Let $m > n$, and let $A(z) \in \mathbf{U}$. Determine an integer $\delta \geq 1$ such that

$$A(z) = A_1(z) + z^{m-n+\delta}A_2(z), \tag{3.57}$$

where

$$A_1(z) = A(z) \ mod \ z^{m-n+1} \tag{3.58}$$

and $A_2(0) \neq 0$ if $\delta < \infty$. If $\delta \leq n$, let $S_{n,n-\delta}(z) / T_{n,n-\delta}(z)$ be the scaled Pade fraction of type $(n, n-\delta)$ for $1/A_2(z)$. Then the scaled Pade fraction $S_{mn}(z) / T_{mn}(z)$ of type (m,n) for $A(z)$ is given by

$$S_{mn}(z) = \begin{cases} A_1(z) S_{n,n-\delta}(z) + z^{m-n+\delta} T_{n,n-\delta}(z), & \text{if } \delta \leq n \\ A_1(z)\, z^n, & \text{otherwise,} \end{cases} \tag{3.59}$$

$$T_{mn}(z) = \begin{cases} S_{n,n-\delta}(z)\,, & \text{if } \delta \leq n \\ z^n\,, & \text{otherwise.} \end{cases} \tag{3.60}$$

Proof: If $\delta \leq n$, then

$$\begin{aligned} \partial(S_{mn}) &= \partial(A_1 S_{n,n-\delta} + z^{m-n+\delta} T_{n,n-\delta}) \\ &\leq \max\{ (m-n)+n, \ (m-n+\delta)+(n-\delta)\} \\ &= m. \end{aligned}$$

Also, $\partial(T_{mn}) \leq n$. Furthermore, either $\partial(S_{mn}) = m$ and $\partial(T_{mn}) = n$ or both. Moreover,

$$\begin{aligned} &A(z)\cdot T_{mn}(z) - S_{mn}(z) \\ &\quad = \{A_1(z) + z^{m-n+\delta} A_2(z)\}\, S_{n,n-\delta}(z) - \{A_1(z) S_{n,n-\delta}(z) + z^{m-n+\delta} T_{n,n-\delta}(z)\} \\ &\quad = z^{m-n+\delta}\, \{A_2(z) S_{n,n-\delta}(z) - T_{n,n-\delta}(z)\} \\ &\quad = z^{m-n+\delta}\, \{A_2(z) O(z^{2n-\delta+1})\} \\ &\quad = O(z^{m+n+1}). \end{aligned}$$

Finally,

$$\begin{aligned} &GCD(S_{mn}, T_{mn}) \\ &\quad = GCD(A_1 S_{n,n-\delta} + z^{m-n+\delta} T_{n,n-\delta}, S_{n,n-\delta}) \\ &\quad = GCD(z^{m-n+\delta} T_{n,n-\delta}, S_{n,n-\delta}) \\ &\quad = GCD(T_{n,n-\delta}, S_{n,n-\delta}). \end{aligned}$$

If $\delta > n$, then clearly $\partial(S_{mn}) \leq m$ and $\partial(T_{mn}) = n$. Moreover,

$$\begin{aligned} &A(z)\cdot T_{mn}(z) - S_{mn}(z) \\ &\quad = z^n \{A_1(z) + z^{m-n+\delta} A_2(z)\} - z^n A_1(z) \\ &\quad = z^{m+\delta}\, A_2(z) \\ &\quad = O(z^{m+n+1}). \end{aligned}$$

Finally, $GCD(S_{mn}, T_{mn}) = z^n$. ■

Theorem 3.8: For arbitrary $m \geq n$, Algorithm 1 can compute the scaled Pade fraction of type (m, n) for $A(z) \in \mathbf{U}$ in time $O(m \log m) + O(n \log^2 n)$.

Proof: If $\delta > n$ in equation (3.57), the result is trivial.

If $\delta \leq n$, computation of the first $2n - \delta + 1$ terms of $1 / A_2(z)$ requires $O(n \log n)$ operations. Using (3.55),

$$C(n, n-\delta) \leq C(n,n),$$

and consequently by Theorem 3.6, it follows that the cost of computing the scaled Pade fraction $S_{n,n-\delta}(z) / T_{n,n-\delta}(z)$ for $1 / A_2(z)$ is bounded by $O(n \log^2 n)$ operations. Finally, the cost of computing $S_{mn}(z)$ in equation (3.59) is $O(m \log m)$ operations. ■

3.4. Fast Off-diagonal Algorithm for a Quotient Power Series

Let $A(z)$, $B(z)$ be unit power series and let

$$C(z) = -A(z) / B(z) \tag{3.59}$$

be the quotient power series. Given the non-negative integers $m \geq n$, then Algorithm 1 OFF-DIAG can be used to compute the scaled Pade fractions $S_{mn}(z) / T_{mn}(z)$ of type (m,n) for $C(z)$. As a result,

$$A(z)T_{mn}(z) + B(z)S_{mn}(z) = O(z^{m+n+1}) \; . \tag{3.60}$$

Before applying the algorithm, the quotient

$$C(z) \;\; mod \;\; z^{m+n+1} = -A(z) / B(z) \;\; mod \;\; z^{m+n+1} \tag{3.61}$$

must be calculated. However, this division need be computed modulo z^{m+n+1}, only, by modifying

Algorithm 1 as follows:

ALGORITHM 2 : **OFFDIAG**

INPUT: $A(z), B(z), m, n$, where

(1) m and n are non-negative integers with $m \geq n$, and

(2) $A(z), B(z)$ are unit power series . (Note that only $A(z)$ $mod\ z^{m+n+1}$ and $B(z)$ $mod\ z^{m+n+1}$ are required).

OUTPUT: $\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix}$, where

(1) S_1 / T_1 is the scaled Pade fraction of type (m, n) for $-A(z) / B(z)$,and

(2) S_0 / T_0 is the predecessor scaled Pade fraction of type $(m-\lambda_{mn}-1, n-\lambda_{mn}-1)$ for $-A(z) / B(z)$, given that

$$z^{\lambda_{mn}} = GCD(S_1,T_1) .$$

Step 1: # Initialization #

$$i \leftarrow -1$$
$$M \leftarrow (m-n)$$
$$N \leftarrow 0$$

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} -A(z) / B(z)\ (mod\ z^{M+1}) & z^{M-1} \\ 1 & 0 \end{bmatrix}$$

Step 2: #Calculation of step-size #

$$i \leftarrow i + 1$$
$$s \leftarrow \min \{2^i - N , n - N\}$$

Step 3: # Termination criterion #

If $s = 0$ then exit

Step 4: # Calculation of scaling factor for S_1 / T_1 #

Determine λ_{MN} such that

$$z^{\lambda_{MN}} = GCD(S_1, T_1)$$

Step 5: # Computation of residual for $\gamma_{MN}(z) = S_1 / T_1$ #

Compute μ_{MN} and R_1 such that

$$(AT_1 + BS_1) \bmod z^{M+N+2s+\lambda_{MN}+1} = z^{M+N+\mu_{MN}+1} R_1 ,$$

where $R_1(0) \neq 0$ if $\mu_{MN} < 2s + \lambda_{MN}$.

Step 6: # Identification of Cases #

if $\mu_{MN} \geq s$

then # Case of Theorem 3.1 #

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \begin{bmatrix} z^s & 0 \\ 0 & 1 \end{bmatrix}$$

go to step 11

else # Case of Theorem 3.4 and 3.5 #

go to step 7

Step 7: # Calculation of degrees for residual scaled Pade fractions #

$$\bar{m} \leftarrow s + \lambda_{MN}$$
$$\bar{n} \leftarrow s - \mu_{MN} - 1$$

Step 8: # Computation of residual for $\gamma_{M^*N^*}(z) = S_0 / T_0$ #

Compute R_0 such that

$$(A \cdot T_0 + B \cdot S_0) \ mod \ z^{M+N+\bar{m}+\bar{n}-2\lambda_{MN}} = R_0 \, z^{M+N-2\lambda_{MN}-1}$$

where $R_0(0) \neq 0$.

Step 9: # Computation of residual scaled Pade fractions #

$$\begin{bmatrix} \bar{S}_1 & \bar{S}_0 \\ \bar{T}_1 & \bar{T}_0 \end{bmatrix} \leftarrow \textbf{OFFDIAG} \ (R_0, R_1, \bar{m}, \bar{n})$$

Step 10: # Advancement of scaled Pade fraction computation #

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \begin{bmatrix} z^{-\lambda_{MN}} & 0 \\ 0 & z^{\lambda_{MN}+\mu_{MN}+2} \end{bmatrix} \begin{bmatrix} \bar{S}_1 & \bar{S}_0 \\ \bar{T}_1 & \bar{T}_0 \end{bmatrix}$$

Step 11: # Calculation of degrees of S_1 / T_1 #

$M \leftarrow M + s$
$N \leftarrow N + s$
goto step 2 ■

Algorithm 2 is a generalization of Algorithm 1, since it can be used to produce scaled Pade fractions for a single power series $A(z)$ simply be setting $B(z) = -1$. It differs from Algorithm 1 in that the division

$$\bar{A}(z) = -R_0(z) / R_1(z) \ mod \ z^{\bar{m}+\bar{n}+1} \tag{3.63}$$

in step 9 of Algorithm 1 is avoided. Instead, the division is delayed (i.e., immediately subsequent to, rather than prior to, the recursive call of OFFDIAG) until the initialization

$$S_1 = -A(z) / B(z) \;\; mod \;\; z^{\bar{m} - \bar{n} + 1} \tag{3.64}$$

in step 1 of Algorithm 2.

There are also various sign changes introduced in steps 1, 5, 8 and 10. These account for the fact that the algorithm deals with the power series $-A(z) / B(z)$ rather than $A(z) / B(z)$. This notational change simplifies the development of subsequent results.

For implementation purposes, Algorithm 2 can result in considerable savings in cost. Practically, fast division is significantly slower than fast multiplication, by an asymptotic constant of approximately 7 (see Verheijen [VER83]). For example, if $-A(z) / B(z)$ is normal, then $\bar{m} - \bar{n} + 1 = 2$ and the division in step 1 of Algorithm 2 becomes trivial. However, the asymptotic cost of Algorithm 2 remains the same as the asymptotic cost of Algorithm 1 given in section 3.3.

The proof of the correctness of Algorithm 2 is nearly identical to the proof of correctness of Algorithm 1, and therefore it is not given.

3.5. Classical Off-diagonal Algorithm for a Quotient Power Series

Let S_1 / T_1 be the scaled Pade fraction of type (M,N) for $-A(z)/B(z)$ such that

$$z^{\lambda_{MN}} = GCD(S_1 , T_1) = 1.$$

That is, $\lambda_{MN} = 0$. Then

$$A \cdot T_1 + B \cdot S_1 = z^{M+N+\mu_{MN}+1} R_1 ,$$

where $R_1(0) \neq 0$ if $\mu_{MN} < \infty$. Thus, $(z^k S_1) / (z^k T_1)$ is the scaled Pade fraction of type $(M+k , N+k)$ for $-A(z)/B(z)$ whenever $0 \leq k \leq \mu_{MN}$. If $\mu_{MN} < \infty$, the next distinct Pade fraction for $-A(z) / B(z)$ along the $(M-N)-th$ off-diagonal path is of type $(M+s , N+s)$, where

$$s = \mu_{mn} + 1 \ .$$

By so selecting the step-size s, Algorithm 2 may be used to compute all Pade fractions along the $(M-N)-th$ off-diagonal path. With this choice of s, step 7 in Algorithm 2 gives

$$\bar{m} = \mu_{MN} + 1$$

$$\bar{n} = 0 \quad .$$

Consequently, the recursive call of OFFDIAG in step 10 reduces to

$$\begin{bmatrix} \bar{S}_1 & \bar{S}_0 \\ \bar{T}_1 & \bar{T}_0 \end{bmatrix} \leftarrow \begin{bmatrix} -R_0/R_1 \ mod \ z^{\mu_{MN}+2} & z^{\mu_{MN}} \\ 1 & 0 \end{bmatrix},$$

and step 11 then yields

$$\begin{bmatrix} S_1 & S_0 \\ T_1 & T_0 \end{bmatrix} \leftarrow \begin{bmatrix} S_1 \cdot \bar{S}_1 + z^{\mu_{MN}+2} \cdot S_0 & z^{\mu_{MN}} \cdot S_1 \\ T_1 \cdot \bar{S}_1 + z^{\mu_{MN}+2} \cdot T_0 & z^{\mu_{MN}} \cdot T_1 \end{bmatrix}.$$

Since $GCD(\bar{S}_1 , \bar{T}_1) = 1$, it follows from the proof of theorem 3.4 that

$$GCD \ (S_1 \cdot \bar{S}_1 + z^{\mu_{MN}+2} \cdot S_0 \quad , T_1 \cdot \bar{S}_1 + z^{\mu_{MN}+2} \cdot T_0) = 1 \ .$$

Thus, the above computations can be repeated for the scaled Pade fraction of type $(M + s , N + s)$.

The full details are provided in Algorithm 3, below. To simplify the presentation of subsequent results, at the i-th iteration, the scaled Pade fractions S_1 / T_1 of type (M,N) is denoted by S_i / T_i and its predecessor S_0 / T_0 by S_{i-1} / T_{i-1}. In addition, the residual power series R_1 and R_0 are denoted by R_i and R_{i-1}, respectively.

ALGORITHM 3: **OFFDIAG**

INPUT: A, B, m, n , where

(1) m and n are non-negative integers with $m \geq n$, and

(2) A and B are unit power series . (Note that only $A \ mod\ z^{m+n+1}$ and $B \ mod\ z^{m+n+1}$ are required).

OUTPUT: $\begin{bmatrix} S_{i+1} & S_i \\ T_{i+1} & T_i \end{bmatrix}$, where

(1) S_{i+1} / T_{i+1} is the scaled Pade fraction of type (m, n) for $-A/B$, and

(2) S_i / T_i is the scaled Pade fraction of type $(m-\lambda_{mn}-1, n-\lambda_{mn}-1)$ for $-A/B$, given that

$$z^{\lambda_{mn}} = GCD(S_{i+1}, T_{i+1}) \ .$$

Step 1: # Initialization #

$$i \leftarrow -1$$
$$M \leftarrow (m-n)$$
$$N \leftarrow 0$$

$$\begin{bmatrix} S_{i+1} & S_i \\ T_{i+1} & T_i \end{bmatrix} \leftarrow \begin{bmatrix} -A(z)/B(z) \ mod\ z^{M+1} & z^{M-1} \\ 1 & 0 \end{bmatrix}$$

Step 2: # Termination criterion #

If $N = n$
then exit
else $i \leftarrow i + 1$

Step 3: # Computation of residual for $\gamma_{MN} = S_i / T_i$ #

Compute μ_{MN} and R_i such that

$$(A \cdot T_i + B \cdot S_i) \ mod\ z^{M+N+2\mu_{MN}+3} = z^{M+N+\mu_{MN}+1} \ R_i \ ,$$

where $R_i(0) \neq 0$ if $\mu_{MN} < 2\,(n - N)$.

Step 4: # Calculation of step-size #

$$s \leftarrow \min\{\, \mu_{MN} + 1 \,,\ n - N \,\}$$

Step 5: # Identification of Cases #

if $\mu_{MN} \geq s$

then # Case of Theorem 3.1 #

$$\begin{bmatrix} S_{i+1} & S_i \\ T_{i+1} & T_i \end{bmatrix} \leftarrow \begin{bmatrix} S_i & S_{i-1} \\ T_i & T_{i-1} \end{bmatrix} \begin{bmatrix} z^s & 0 \\ 0 & 1 \end{bmatrix}$$

go to step 8

else # Case of Theorem 3.4 and 3.5 #

go to step 6

Step 6: # Computation of residual for $\gamma_{M^* N^*}(z) = S_{i-1} / T_{i-1}$ #

Compute R_{i-1} such that

$$(A \cdot T_{i-1} + B \cdot S_{i-1}) \ mod\ z^{M+N+\mu_{MN}+1} = z^{M+N-1} R_{i-1} \,,$$

where $R_{i-1}(0) \neq 0$.

Step 7: # Advancement of scaled Pade fraction computation #

$$\begin{bmatrix} S_{i+1} & S_i \\ T_{i+1} & T_i \end{bmatrix} \leftarrow \begin{bmatrix} S_i & S_{i-1} \\ T_i & T_{i-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & z^{\mu_{MN}+2} \end{bmatrix} \begin{bmatrix} -R_{i-1} / R_i \ mod\ z^{\mu_{MN}+2} & z^{\mu_{MN}} \\ 1 & 0 \end{bmatrix}$$

Step 8: # Calculation of degrees of S_{i+1} / T_{i+1} #

$$N \leftarrow N + s$$
$$M \leftarrow M + s$$

go to step 2 ■

In the case that $B(z)=-1$ and $m = n$, Algorithm 3 is precisely the algorithm given by Cabay and Kao [*CAB*83] for computing diagonal Pade fractions for a single power series. Their $O(n^2)$ algorithm (since steps 3 and 6 require only $\mu_{MN} + 2$ terms of a product polynomials, and since one of the multipliers in step 7 is a polynomial of degree $\mu_{MN} + 1$, for small μ_{MN}, no advantage can be gained from using fast methods for polynomial arithmetic) is shown to be faster than other $O(n^2)$ algorithms for computing diagonal Pade fractions, such as those of Trench [*TRE*65] and Rissanen [*RIS*73].

The more interesting observation about Algorithm 3 is made in the next chapter, however. It is shown that, when the given quotient power series is a rational function, Algorithm 3 along one specific off-diagonal path corresponds exactly to Euclid's extended algorithm for computing the greatest common divisor of the numerator and denominator of the given rational function. In this sense, Algorithm 3 is a generalization of Euclid's extended algorithm. It is for this reason that we choose to call Algorithm 3 classical.

CHAPTER 4

GREATEST COMMON DIVISOR COMPUTATIONS OF POLYNOMIALS

4.1. Introduction

In this chapter, Algorithm 2 and Algorithm 3 are examined when they are applied to $-A(z)/B(z)$, where $A(z)$ and $B(z)$ are polynomials of degrees m and n respectively. In addition, it is assumed that $A(0)\neq 0$ and $B(0)\neq 0$.

To permit the analysis, we need the following

Definition: The reciprocal of a polynomial

$$P(z) = p_0 + p_1 z + \cdots + p_n z^n \tag{4.1}$$

of at most degree n is defined to be

$$P^R(z) = p_0 z^n + p_1 z^{n-1} + \cdots + p_n = z^n P(z^{-1}). \tag{4.2}$$

The name originates from the fact that the zeros of $P^R(z)$ are the reciprocals of those of $P(z)$ if $p_0 p_n \neq 0$.

Clearly

$$[P^R(z)]^R = P(z) \ ; \tag{4.3}$$

that is, the operation of forming the reciprocal polynomial is involuntary. Also

$$[P(z) \cdot Q(z)]^R = P^R(z) \cdot Q^R(z), \tag{4.4}$$

and

$$[z^\lambda P(z)]^R = P^R(z). \tag{4.5}$$

Let $A(z)$ be a unit power series. The first $n+1$ terms of $A(z)$ are denoted by

$$A_n(z) = \sum_{i=0}^{n} a_i z^i$$

$$= A(z) \mod z^{n+1}. \quad (4.6)$$

The reciprocal $A_n^R(z)$ of $A_n(z)$ is then given by

$$A_n^R(z) = \sum_{i=0}^{n} a_{n-i} z^i. \quad (4.7)$$

4.2. Euclid's Extended Algorithm

Theorem 4.1. Algorithm 3 applied to $-A(z)/B(z)$, where $A(z)$ and $B(z)$ are polynomials of degree m and n respectively, is equivalent to Euclid's extended algorithm for computing the greatest common divisior of $A^R(z)$ and $B^R(z)$.

Proof: For completeness in presentation, assume that modulo operations are not performed in steps 3 and 6 of Algorithm 3. Thus,

$$A \cdot T_i + B \cdot S_i = z^{M+N+\mu_{MN}+1} R_i \quad (4.8)$$

$$A \cdot T_{i-1} + B \cdot S_{i-1} = z^{M+N-1} R_{i-1}, \quad (4.9)$$

where $R_i(z)$ and $R_{i-1}(z)$ are polynomials of degrees $n-N-\mu_{MN}-1$ and $n-N$, respectively [1]. By taking reciprocals of (4.8) and (4.9), it follows that

$$A^R \cdot T_i^R + B^R \cdot S_i^R = R_i^R \quad (4.10)$$

and

$$A^R \cdot T_{i-1}^R + B^R \cdot S_{i-1}^R = R_{i-1}^R. \quad (4.11)$$

---

[1]By convention, a polynomial of negative degree is the zero polynomial.

Let

$$Q_{i+1}(z) = R_{i-1}(z) \,/\, R_i(z) \quad mod \;\; z^{\mu_{MN}+2} \tag{4.12}$$

be a polynomial of degree $\mu_{MN}+1$. Then

$$R_{i-1}(z) - Q_{i+1}(z) \cdot R_i(z) = z^{\mu_{MN}+2} \; \bar{R}(z), \tag{4.13}$$

for some polynomial $\bar{R}(z)$ of degree $m-N-\mu_{MN}-2$. Taking reciprocals of (4.13), it follows that

$$R_{i-1}^R(z) - Q_{i+1}^R(z) \cdot R_i^R(z) = \bar{R}^R(z). \tag{4.14}$$

That is, $Q_{i+1}^R(z)$ and $\bar{R}^R(z)$ are the quotient and remainder, respectively, on division of $R_{i-1}^R(z)$ by $R_i^R(z)$.

With $Q_{i+1}(z)$ defined by (4.12), step 7 of Algorithm 3 yields

$$\begin{bmatrix} S_{i+1} & S_i \\ T_{i+1} & T_i \end{bmatrix} \leftarrow \begin{bmatrix} z^{\mu_{MN}+2} \cdot S_{i-1} - Q_{i+1} \cdot S_i & z^{\mu_{MN}} \cdot S_i \\ z^{\mu_{MN}+2} \cdot T_{i-1} - Q_{i+1} \cdot T_i & z^{\mu_{MN}} \cdot T_i \end{bmatrix} , \tag{4.15}$$

which on taking reciprocals becomes

$$\begin{bmatrix} S_{i+1}^R & S_i^R \\ T_{i+1}^R & T_i^R \end{bmatrix} \leftarrow \begin{bmatrix} S_{i-1}^R - Q_{i+1}^R \cdot S_i^R & S_i^R \\ T_{i-1}^R - Q_{i+1}^R \cdot T_i^R & T_i^R \end{bmatrix}. \tag{4.16}$$

Looking ahead one iteration, we obtain

$$\begin{aligned} R_{i+1}^R &= A^R \cdot T_{i+1}^R + B^R \cdot S_{i+1}^R \\ &= A^R \cdot [\, T_{i-1}^R - Q_{i+1}^R \cdot T_i^R \,] + B^R \cdot [\, S_{i-1}^R - Q_{i+1}^R \cdot S_i^R \,] \\ &= R_{i-1}^R - Q_{i+1}^R \cdot R_i^R. \end{aligned} \tag{4.17}$$

Thus, $\bar{R}^R(z) = R_{i+1}^R(z)$ in equation (4.14).

Summarizing, let Q_{i+1}^R be the quotient on division of R_{i-1}^R by R_i^R. Then

$$\begin{aligned} R_{i+1}^R &= R_{i-1}^R - Q_{i+1}^R \cdot R_i^R \\ S_{i+1}^R &= S_{i-1}^R - Q_{i+1}^R \cdot S_i^R \end{aligned} \tag{4.18}$$

$$T_{i+1}^R = T_{i-1}^R - Q_{i+1}^R \cdot T_i^R ,$$

which are the fundamental relations describing Euclid's extended algorithm.

To complete the proof, it remains to show that the initial conditions for Euclid's extended algorithm are satisfied. Initialization in step 1 of Algorithm 3 yields

$$\begin{bmatrix} S_0 & S_{-1} \\ T_0 & T_{-1} \end{bmatrix} = \begin{bmatrix} -Q_0 & z^{m-n-1} \\ 1 & 0 \end{bmatrix} , \tag{4.19}$$

where, as in (4.12), Q_0^R is the quotient on division of A^R by B^R. Steps 3 and 5 with $i = 0$ then become

$$A \cdot 1 + B \cdot (-Q_0) = z^{m-n+\mu_{m-n,0}+1} R_0 \tag{4.20}$$

and

$$A \cdot 0 + B \cdot z^{m-n-1} = z^{m-n-1} R_{-1} \tag{4.21}$$

Taking reciproicals of (4.19), (4.20) and (4.21), it follows that

$$\begin{array}{lll} R_{-1}^R = B^R & S_{-1}^R = 1 & T_{-1}^R = 0 \\ R_0^R = A^R - Q_0^R \cdot B^R & S_0^R = -Q_0^R & T_0^R = 1 \end{array} . \tag{4.22}$$ ■

Corollary 4.2. At the i-th iteration of Algorithm 3, let S_i / T_i be the scaled Pade fraction of type (M,N) for $-A / B$, such that

$$A \cdot T_i + B \cdot S_i = z^{M+N+\mu_{MN}+1} R_i .$$

Then,

$$\begin{aligned} R_{i-1} \cdot S_i - z^{\mu_{MN}+2} R_i \cdot S_{i-1} &= (-1)^i \cdot A \\ R_{i-1} \cdot T_i - z^{\mu_{MN}+2} R_i \cdot T_{i-1} &= (-1)^{i+1} \cdot B \\ S_{i-1} \cdot T_i - S_i \cdot T_{i-1} &= (-1)^{i+1} z^{M+N-1} . \end{aligned} \tag{4.23}$$

Furthermore, the algorithm terminates for some $i = k$, where $k<n$. On termination, S_{k+1} / T_{k+1} is

the scaled Pade fraction of type (m,n) for $-A/B$ such that

$$A \cdot T_{k+1} + B \cdot S_{k+1} = 0. \tag{4.24}$$

In addition,

$$A \cdot T_k + B \cdot S_k = z^{m+n-2\lambda_{mn}-1} R_k, \tag{4.25}$$

where

$$z^{\lambda_{mn}} = GCD(S_{k+1}, T_{k+1})$$

and

$$R_k = GCD(A, B).$$

Proof: From Euclid's extended algorithm (4.18) with initial conditions (4.22), it follows that (see, for example, McEliece [$McE78$])

$$\begin{aligned} R^R_{i-1} \cdot S^R_i - R^R_i \cdot S^R_{i-1} &= (-1)^i \cdot A^R \\ R^R_{i-1} \cdot T^R_i - R^R_i \cdot T^R_{i-1} &= (-1)^{i+1} \cdot B^R \\ S^R_{i-1} \cdot T^R_i - S^R_i \cdot T^R_{i-1} &= (-1)^{i+1}. \end{aligned} \tag{4.26}$$

By using the correspondence established in theorem 4.1 and taking reciprocals, equations (4.26) result in equations (4.23).

Furthermore, Euclid's extended algorithm terminates for some $k < n$ when

$$A^R \cdot T^R_{k+1} + B^R \cdot S^R_{k+1} = R^R_{k+1} = 0 \tag{4.27}$$

and

$$A^R \cdot T^R_k + B^R \cdot S^R_k = R^R_k = GCD\,(A^R, B^R), \tag{4.28}$$

where $\lambda = \partial(R^R_k)$, $\partial(S^R_{k+1}) \le m - \lambda$, $\partial(T^R_{k+1}) \le n - \lambda$, $\partial(S^R_k) < m - \lambda$ and $\partial(T^R_k) < n - \lambda$.

Taking reciprocals of (4.28) with respect to $z^{m+n-\lambda-1}$ results in

$$A \cdot T_k + B \cdot S_k = z^{m+n-\lambda-1} \cdot R_k ,$$

where $R_k(0) \neq 0$. Thus, S_k / T_k is the scaled Pade fraction of type $(m-\lambda-1, n-\lambda-1)$ for $-A / B$. Taking reciprocals of (4.27) with respect to z^{m+n} gives

$$A \cdot T_{k+1} + B \cdot S_{k+1} = 0, \tag{4.29}$$

where $\partial(S_{k+1}) \leq m$ and $\partial(T_{k+1}) \leq n$. Thus, S_{k+1} / T_{k+1} is the scaled Pade fraction of type (m,n) such that

$$GCD(S_{k+1}, T_{k+1}) = z^{\lambda} . \tag{4.30}$$

Thus $\lambda = \lambda_{mn}$. ■

As a consequence of Corollary 4.2. Algorithm 3 computes co-multipliers S_k and T_k, only, such that

$$A^R \cdot T_k^R + B^R \cdot S_k^R = R_k^R = GCD(A^R, B^R) . \tag{4.31}$$

The remainder R_k^R is available only if the multiplications in steps 3 and 6 are performed without the modulo operation.

4.3. Fast GCD Computations

Theorem 4.3. Algorithm 2 applied to $-A(z) / B(z)$, where $A(z)$ and $B(z)$ are polynomials of degree m and n, respectively, returns the scaled Pade fraction S_1 / T_1 of type (m,n) such that

$$A^R \cdot T_1^R + B^R \cdot S_1^R = 0 \tag{4.32}$$

and its predecessor S_0 / T_0 of type $(m-\lambda_{mn}-1, n-\lambda_{mn}-1)$ such that

$$A^R \cdot T_0^R + B^R \cdot S_0^R = R_0^R , \tag{4.33}$$

where

$$z^{\lambda_{mn}} = GCD(S_1\ ,\ T_1)$$

and

$$R_0 = GCD(A\ ,\ B).$$

Proof: Since scaled Pade fractions are unique, the result is an immediate consequence of Corollary 4.2. ∎

The greatest common divisor R_0 is not explicitly computed by Algorithm 2. However, $\partial(R_0) = \lambda_{m\,n} \leq n$. Using fast multiplication, it can therefore be determined in $O(n \log n)$. As a consequence of this and Theorem 3.8, we have

Theorem 4.4. Algorithm 2 can compute the greatest common divisor, the cofactors and the comultipliers of two polynomials of degrees m and n, where $m \geq n$, in time $O(m \log m) + O(n \log^2 n)$. ∎

Thus, Algorithm 2 for GCD computations is basically of the same asymptotic complexity as the fast algorithm of Moenck [*MOE*73], Aho, Hopcroft and Ullman [*AHO*74] and Brent, Gustavson and Yun [*BRE*80], which are of complexity $O(\ (m+n) \log^2 (m+n)\ )$. However, Algorithm 2 has the advantage of being partly iterative (approximately half as many recursive calls are used in comparison with the other fast methods). This can result in significant cost savings in an implementation environment (Verheijen [*VER*83]).

4.4. Antidiagonal Computations

Let

$$A(z) = \sum_{i=0}^{\infty} a_i\, z^i \tag{4.34}$$

be a unit power series, and let

$$A_d^R(z) = \sum_{i=0}^{d} a_{d-i}\, z^i \quad , \quad d \geq 0, \tag{4.35}$$

be the reciprocal of the first $d+1$ terms of $A(z)$. In this section, we examine the intermediate results obtained by Algorithm 2 [2] while computing the scaled Pade fraction of type (n,n), $2n \leq d$, for the two polynomials $1 + z\, A_d^R(z)$ and $B^R(z) = -1$.

At the i-th iteration of Algorithm 2, the scaled Pade fraction $S_{NN}(z) \,/\, T_{NN}(z)$ of type (N,N) for $-\{1+z\, A_d^R(z)\} \,/\, B^R(z)$ and its predecessor $S_{N^*N^*}(z) \,/\, T_{N^*N^*}(z)$ are determined such that

$$\{\, 1+z\, A_d^R(z)\} \cdot T_{NN}(z) - S_{NN}(z) = z^{2N+\mu_{NN}+1}\; R_1(z) \tag{4.36}$$

$$\{1+ z\, A_d^R(z)\} \cdot T_{N^*N^*}(z) - S_{N^*N^*}(z) = z^{2N^*+1}\; R_0(z)\ , \tag{4.37}$$

where

$$N^* = N - \lambda_{NN} - 1$$

and

$$z^{\lambda_{NN}} = GCD(S_{NN}, T_{NN})\ .$$

By taking reciprocals of (4.36) and (4.37), it follows that

$$A_d(z) \cdot T_{NN}^R(z) - R_1^R(z) = z^{d+1}\; \{S_{NN}^R(z) - T_{NN}^R(z)\} \tag{4.38}$$

$$A_d(z) \cdot T_{N^*N^*}^R(z) - R_0^R(z) = z^{d+1}\; \{S_{N^*N^*}^R(z) - T_{N^*N^*}^R(z)\} \tag{4.39}$$

where

$$\partial(R_1^R) = d - N - \mu_{NN} \tag{4.40}$$

and

$$\partial(R_0^R) = d - N^*. \tag{4.41}$$

---

[2] It is assumed that no truncation of polynomial operations takes place.

Theorem 4.5: Let $M=d-N$ and $M^* = d - N^*$. Then the scaled Pade fractions of type (M, N) and (M^*, N^*) for $A(z)$ are $R_1^R(z) / T_{NN}^R(z)$ and $R_0^R(z) / T_{N^*N^*}^R(z)$, respectively.

Proof: From (4.38), (4.39), (4.40) and (4.41), clearly the degree and order conditions for scaled Pade fractions are satisfied. To complete the proof, let

$$G(z) = GCD(T_{NN}^R, R_1^R).$$

Then, from (4.38), $G(z)$ must divide not only T_{NN}^R but $z^{d+1} S_{NN}^R(z)$, as well. Since $GCD(S_{NN}^R, T_{NN}^R) = 1$, it follows that

$$G(z) = z^\lambda$$

for some $\lambda \geq 0$. Thus, $R_1^R(z) / T_{NN}^R(z)$ is a scaled Pade fraction of type (M,N). Similarly, $R_0^R(z) / T_{N^*N^*}^R(z)$ is a scaled Pade fraction of type (M^*, N^*). ∎

Observe that

$$M + N = d \tag{4.42}$$

and

$$M^* + N^* = d \ . \tag{4.43}$$

The scaled Pade fractions $R_1^R(z) / T_{NN}^R(z)$ and $R_0^R(z) / T_{N^*N^*}^R(z)$ both lie along the d-th anti-diagonal path of the scaled Pade table $\Gamma(A)$, where

Definition (see, for example, Brent [*BRE*80]). The d-th anti-diagonal path of the scaled Pade table $\Gamma(A)$ for a unit power series $A(z)$ is defined to the set of all scaled Pade fractions of type (i,j), where

$$i + j = d \ . \tag{4.44}$$

But, by Theorem 4.5, the scaled Pade fraction $R_1^R(z) / T_{NN}^R(z)$ and its predecessor $R_0^R(z) / T_{N^*N^*}^R(z)$, along the d-th anti-diagonal path, are obtained by applying Euclid's extended

algorithm to the reciprocals of $1 + z\,A_d^R(z)$ and $B^R(z) = -1$, that is, to the polynomials $A_d(z) + z^{d+1}$ and $-z^{d+1}$. However,

$$GCD(\,A_d(z) + z^{d+1}\,,\,-z^{d+1}\,) = GCD(\,A_d(z)\,,\,-z^{d+1}\,). \tag{4.45}$$

This is precisely the result given by McElice [*McE*78]; namely, Euclid's extended algorithm applied to $A_d(z)$ and $-z^{d+1}$ yields all the Pade fractions along the d-th anti-diagonal path of the Pade table for $A(z)$. Equivalently, by Theorem 4.5, the same Pade fractions can be obtained by applying Algorithm 3 to the two polynomials $1 + z\,A_d^R(z)$ and $B^R(z) = -1$, and then taking reciprocals of the results.

In addition, by Theorem 4.5, Algorithm 2 can be used to compute any specific scaled Pade fraction $R_1^R(z)\,/\,T_{NN}^R(z)$ of type $(M\;,N)$, where $M + N = d$, along the d-th anti-diagonal path for $A(z)$. This requires $O(\,N\log^2 N)$ arithmetic operations to compute, by Algorithm 2, the scaled Pade fraction $S_{NN}(z)\,/\,T_{NN}(z)$ of type (N,N) for $1 + z\,A_d^R(z)$, plus an additional $O(\,(d-N)\log\,(d-N)\,)$ operations to determine $R_1(z)$ which satisfies (4.36). This asymptotic cost is the same as the cost of the fast algorithm of Brent [*BRE*80] for computing the greatest common divisor of $A_d(z)$ and $-z^{d+1}$. However, their PRSDC algorithm can compute a specific Pade fraction along the d-th anti-diagonal only with significantly more complications.

CHAPTER 5

THE INVERSE OF HANKEL AND TOEPLITZ MATRICES

5.1. Introduction

In this chapter, scaled Pade fractions are used to construct the inverse of a nonsingular Hankel matrix

$$H_{nn} = \begin{bmatrix} a_1 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-1} \end{bmatrix}$$

and the inverse of the Toeplitz matrix

$$T_{nn} = \begin{bmatrix} a_{2n-1} & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_1 \end{bmatrix}$$

associated with the Hankel matrices.

The formula derived for the inverse is similar to those given by Trench [*TRE*65], Zohar [*ZOH*74], and Kailath et al [*KAI*78], but it has the advantage of being successful in the degenerate case when a principal minor of H_{nn} is zero. The formula given by Gohberg and Semencul [*BRE*80] does succeed for the degenerate case, however, it is a slower, $O(n^2)$, algorithm.

5.2. The Inverse Formula

Let

$$S(z)/T(z) = \sum_{i=0}^{n} s_i\, z^i \,/\, \sum_{i=0}^{n} t_i\, z^i \tag{5.1}$$

be the scaled Pade fraction of type (n,n) for

$$A(z) = \sum_{i=0}^{\infty} a_i\, z^i \ . \tag{5.2}$$

Assume that

$$a_{2n} = 0 \ . \tag{5.3}$$

Since $S(z)/T(z)$ is also a Pade form of type (n,n), the denominator $T(z)$ satisfies

$$\begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{n-1} & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & \cdots & a_{2n-1} & 0 \end{bmatrix} \begin{bmatrix} t_n \\ \cdot \\ \cdot \\ \cdot \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} . \tag{5.4}$$

Furthermore, from Collorary 2.8, $\det(H_{nn}) \neq 0$ if only if $t_0 \neq 0$. In the remainder of this chapter, assume $\det(H_{nn}) \neq 0$. We now shall provide an algorithm for computing the inverse of H_{nn}.

Let

$$U(z) \,/\, V(z) = \sum_{i=0}^{n-1} u_i\, z^i \,/\, \sum_{i=0}^{n-1} v_i\, z^i \tag{5.5}$$

be the scaled Pade fraction of type $(n-1,n-1)$ for $A(z)$ (the predecessor of $S(z) \,/\, T(z)$). Then $V(z)$ satisfies

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n-1} & \cdots & a_{2n-2} \end{bmatrix} \begin{bmatrix} v_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ v_1 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{5.6}$$

Lemma 5.1. Let

$$\gamma = (a_n, \cdots, a_{2n-1}) \cdot (v_{n-1}, \cdots, v_0)^t. \tag{5.7}$$

Then $\gamma \neq 0$.

Proof: Suppose $\gamma = 0$. Then, from (5.6), $(v_{n-1}, \cdots, v_0, 0)^t$ satisfies

$$\begin{bmatrix} a_1 & \cdots & a_n & a_{n+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_{n-1} & \cdots & a_{2n-2} & a_{2n-1} \\ a_n & \cdots & a_{2n-1} & 0 \end{bmatrix} \begin{bmatrix} v_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ v_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix};$$

that is, $(v_{n-1}, \cdots, v_0, 0)^t$ is also a solution of (5.4). This implies $\det(H_{nn}) = 0$, which is a contradiction. ∎

Lemma 5.2: Let $B_{nn} = (b_{ij})_{i,j=1}^n$ denote the inverse of H_{nn}. Then

$$b_{i,n} = b_{n,i} = v_{n-i} / \gamma \quad, i = 1, \cdots, n, \tag{5.8}$$

where γ is given by (5.7).

Proof: Together with γ, system (5.6) becomes

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n-1} & \cdots & a_{2n-2} \\ a_n & \cdots & a_{2n-1} \end{bmatrix} \begin{bmatrix} v_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \gamma \end{bmatrix}.$$

Equivalently,

$$\begin{bmatrix} v_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ v_0 \end{bmatrix} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ \gamma \end{bmatrix},$$

and (5.8) now follows since B_{nn} is symmetric. ■

To obtain the remaining elements of B_{nn}, two cases, $t_n \neq 0$ and $t_n = 0$, of the solution of (5.4) are identified.

Case 1: Assume that $t_n \neq 0$. Then (5.4) becomes

$$t_n \cdot \begin{bmatrix} a_1 \\ \cdot \\ \cdot \\ \cdot \\ a_{n-1} \\ a_n \end{bmatrix} + \begin{bmatrix} a_2 & \cdots & a_n & a_{n+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & \cdots & a_{2n-1} & 0 \end{bmatrix} \cdot \begin{bmatrix} t_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 0 \end{bmatrix}. \tag{5.9}$$

Let

$$H_{n+1,n} = \begin{bmatrix} a_2 & \cdots & a_n & a_{n+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_n & \cdots & a_{2n-2} & a_{2n-1} \\ a_{n+1} & \cdots & a_{2n-1} & 0 \end{bmatrix}. \tag{5.10}$$

Then $\det(H_{n+1,n}) \neq 0$, for otherwise (5.9) also has a solution with $t_n = 0$. This is a contradiction of the uniqueness (up to a scalar) of the solution of (5.4).

Lemma 5.3: Assume $t_n \neq 0$, and let $B_{n+1,n} = (\bar{b}_{ij})_{i,j-1}^{n}$ denote the inverse of $H_{n+1,n}$. Then

$$b_{ij} = \bar{b}_{i-1,j} - (t_{n+1-i} / t_0) \bar{b}_{n,j} , \quad i,j = 1, \cdots, n, \tag{5.11}$$

where $\bar{b}_{0,j} = 0, \quad j = 1, \cdots, n.$

Proof: From (5.4)

$$\begin{bmatrix} t_n \\ \cdot \\ \cdot \\ \cdot \\ t_1 \end{bmatrix} [\bar{b}_{n1}, \cdots, \bar{b}_{nn}] = -t_0 \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} \begin{bmatrix} a_{n+1} \\ \cdot \\ \cdot \\ a_{2n-1} \\ 0 \end{bmatrix} [\bar{b}_{n1}, \cdots, \bar{b}_{nn}] . \tag{5.12}$$

However,

$$H_{nn} \begin{bmatrix} 0 & \cdots & 0 \\ \bar{b}_{11} & & \bar{b}_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \bar{b}_{n-1,1} & \cdots & \bar{b}_{n-1,n} \end{bmatrix} = I_n - \begin{bmatrix} a_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{2n-1} \\ 0 \end{bmatrix} [\bar{b}_{n1}, \cdots, \bar{b}_{nn}]. \tag{5.13}$$

Consequently, by substituting (5.13) into (5.12), it follows that

$$\begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ b_{n1} & \cdots & b_{nn} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ \bar{b}_{11} & \cdots & \bar{b}_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \bar{b}_{n-1,1} & \cdots & \bar{b}_{n-1,n} \end{bmatrix}$$

$$-t_0^{-1} \begin{bmatrix} t_n \bar{b}_{n1} & \cdots & t_n \bar{b}_{nn} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ t_1 \bar{b}_{n1} & \cdots & t_1 \bar{b}_{nn} \end{bmatrix} \tag{5.14}$$

and (5.11) follows. ∎

Case 2. Assume $t_n = 0$. Consider the subsystem

$$\begin{bmatrix} a_2 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-2} \end{bmatrix} \begin{bmatrix} t_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ t_1 \end{bmatrix} = -t_0 \begin{bmatrix} a_{n+1} \\ \cdot \\ \cdot \\ \cdot \\ a_{2n-1} \end{bmatrix} \tag{5.15}$$

of (5.4). Then the matrix on the left-hand-side of (5.15) is nonsingular. For, suppose otherwise. Then there exists a nontrivial vector $[\alpha_{n-1}, \cdots, \alpha_1]^t$ such that

$$\begin{bmatrix} a_2 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-2} \end{bmatrix} \begin{bmatrix} \alpha_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \tag{5.16}$$

However, by taking the transpose of (5.15), it follows from (5.16) that

$$\begin{aligned} [a_{n+1}, \cdots, a_{2n-1}] \begin{bmatrix} \alpha_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 \end{bmatrix} &= -t_0^{-1} \, [t_{n-1}, \cdots, t_1] \begin{bmatrix} a_2 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-2} \end{bmatrix} \begin{bmatrix} \alpha_{n-1} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}. \end{aligned} \tag{5.17}$$

Consequently,

$$\begin{bmatrix} a_1 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-1} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha_{n-1} \\ \cdot \\ \cdot \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

which implies that $\det(H_{nn}) \neq 0$. This contradiction implies that

$$\det \begin{bmatrix} a_2 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-2} \end{bmatrix} \neq 0.$$

Now, since $t_n = 0$, it follows from (5.4) that $\det(H_{n+1,n}) = 0$. Let

$$H^*_{n+1,n} = \begin{bmatrix} a_2 & \cdots & a_n & a_{n+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ a_n & & a_{2n-2} & a_{2n-1} \\ a_{n+1} & \cdots & a_{2n-1} & 1 \end{bmatrix}. \qquad (5.18)$$

Then, expansion with respect to the last column yields

$$\det(H^*_{n+1,n}) = \det \begin{bmatrix} a_2 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{n+1} & \cdots & a_{2n-2} \end{bmatrix} \neq 0. \qquad (5.19)$$

Lemma 5.4. Assume $t_n = 0$, and let $B^*_{n+1,n} = (b^*_{ij})^n_{i,j=1}$ denote the inverse of $H^*_{n+1,n}$. Then

$$b_{ij} = b^*_{i-1,j} + (b_{i,n} - t_{n+1-i} / t_0)\, b^*_{n,j} \quad ,i,\ j = 1, \cdots, n, \qquad (5.20)$$

where $b^*_{0,j} = 0$, $j=1, \cdots, n$.

Proof: The result follows using arguments similar to those of the proof of Lemma 5.3. ∎

Theorem 5.5 Let $(t_n, t_{n-1}, \cdots, t_0)'$ and $(v_{n-1}, v_{n-2}, \cdots, v_0)'$ be given by (5.1) and (5.5), respectively. If $t_0 \neq 0$, then the inverse of the H_{nn} is given by the Christoffel-Darboux formula

$$H_{nn}^{-1} = \frac{1}{\gamma\, t_0} \left\{ \begin{bmatrix} t_{n-1} & \cdots & t_0 \\ \vdots & \cdot^{\cdot^{\cdot}} & \\ t_0 & & \mathbf{0} \end{bmatrix} \begin{bmatrix} v_{n-1} & \cdots & v_0 \\ & \ddots & \vdots \\ \mathbf{0} & & v_{n-1} \end{bmatrix} \right.$$

$$\left. - \begin{bmatrix} v_{n-2} & \cdots & v_0 & 0 \\ \vdots & \cdot^{\cdot^{\cdot}} & & \\ v_0 & & \mathbf{0} & \\ 0 & & & \end{bmatrix} \begin{bmatrix} t_n & \cdots & t_1 \\ & \ddots & \vdots \\ \mathbf{0} & & t_n \end{bmatrix} \right\} \tag{5.21}$$

Proof: Consider the case $t_n \neq 0$. Then equation (5.11) of Lemma 5.3 yields

$$\bar{b}_{nn} = -(t_0 / t_n)\, b_{1n}$$

and

$$b_{i+1,n} = \bar{b}_{i,n} - (t_{n-i} / t_0)\, \bar{b}_{nn} \quad , \; i = 1, 2, \cdots, n.$$

Consequently,

$$b_{i+1,n} = \bar{b}_{i,n} + (t_{n-i} / t_n)\, b_{1n} \,, \quad i = 1, 2, \cdots, n. \tag{5.22}$$

Since $\bar{b}_{i,n} = \bar{b}_{n,i}$, equation (5.22) inserted into (5.11) gives

$$b_{ij} = \bar{b}_{i-1,j} - (t_{n+1-i} / t_0)\, [\, b_{j+1,n} - (t_{n-j} / t_n)\, b_{1n} \,] \,.$$

Thus,

$$b_{j+1,i-1} = \bar{b}_{j,i-1} - (t_{n-j} / t_0)\, [\, b_{i,n} - (t_{n+1-i} / t_n)\, b_{1n} \,].$$

Therefore,

$$b_{ij} = b_{j+1,i-1} + t_0^{-1} \; [\, t_{n-j}\, b_{i,n} - t_{n+1-i}\, b_{j+1,n} \,].$$

From (5.8) of Lemma 5.2, it now follows that

$$b_{ij} = b_{j+1,i+1} + (\gamma\, t_0)^{-1} \; [\, t_{n-j}\, v_{n-1} \; - \; t_{n+1-i}\, v_{n-1-j} \,],$$

$$i, j = 1, 2, \cdots, n, \tag{5.23}$$

where all variables with subscripts outside the range $1, 2, \cdots, n$ are assumed to be zero. But equation (5.23) corresponds exactly to (5.21).

The case $t_n = 0$ makes use of Lemma 5.4, and the proof proceeds in a similar manner. ■

Now consider the nonsingular Hankel system

$$H_{n,n}\,\mathbf{x} = \begin{bmatrix} a_1 & \cdots & a_n \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_n & \cdots & a_{2n-1} \end{bmatrix} \mathbf{x} = \mathbf{b}, \tag{5.24}$$

for an arbitrary vector $\mathbf{b}$. From (5.21), the computation of $\mathbf{x} = H_{nn}^{-1} \cdot \mathbf{b}$ can be done by performing four polynomial multiplications. Using fast polynomial multiplication methods, this requires $O(n \log n)$ arithmetic operations. By theorem 3.6, the polynomials $T(z)$ and $V(z)$ can be computed in $O(n \log^2 n)$ operations, and therefore

Theorem 5.6. The system (5.24) can be solved in $O(n \log^2 n)$ arithmetic operations.

Example: Let

$$H_{3,3} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 4 \end{bmatrix},$$

and let

$$A(z) = a_0 + 2\,z + z^4 + 4\,z^5 + 0\,z^6 + \cdots$$

be the associated unit power series. Then the scaled Pade fraction of type (3,3) for $A(z)$ is given by $S(z) \,/\, T(z)$, where

$$S(z) = 2\,a_0 + 4\,(1 - 2a_0) - 16\,(1 - 2a_0)\,z^2 + (64 - a_0)\,z^3$$

and

$$T(z) = 2 - 8z + 32z^2 - z^3 .$$

Thus, $H_{3,3}$ is nonsingular, since $T(0) = t_0 \neq 0$. The predecessor of $S(z) / T(z)$ is the scaled Pade fraction $U(z) / V(z)$ of type (2,2), where

$$U(z) = a_0 z + 2 z^2$$

and

$$V(z) = z.$$

Therefore, the solution of system (5.4),

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} t_3 \\ t_2 \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

is $(t_3, t_2, t_1, t_0)^t = (-1, 32, -8, 2)^t$; and the solution of system (5.6),

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_2 \\ v_1 \\ v_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

is $(v_2, v_1, v_0)^t = (0, 1, 0)$.

The inverse $H_{3,3}$ is obtained by the Christoffel- Darboux formula (5.21) to be

$$H_{3,3}^{-1} = (1/2) \left\{ \begin{bmatrix} 32 & -8 & 2 \\ -8 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 32 & -8 \\ 0 & -1 & 32 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad \blacksquare$$

Since Toeplitz matrices are equivalent to Hankel matrices (simply by a change of notation), the results of theorem 5.6 apply to Toeplitz matrices, as well.

Consequently, by Theorem 5.6, the inverse of a Toeplitz matrix of order n can be obtained in $O(n \log^2 n)$ arithmetic operations. This is done with one call of Algorithm 1 for the computation of a specific scaled Pade fraction. The fast algorithm of Brent [*BRE*80] is also of complexity $O(n \log^2 n)$, but the computation of the Toeplitz inverse may require as many as four calls of their $O(n \log^2 n)$ PRSDC algorithm for computing a specific Pade fraction. In addition, each call is burdened with complications, alluded to earlier in Section 4.4.

CHAPTER 6

CONCLUSIONS AND SUGGESTIONS FOR FUTURE RESEARCH

Central to the classical theory of Pade approximants of power series are the concepts of the Pade form, which always exists but may not be unique, and the Pade fraction, which is unique but in a certain sense may not exist. The fundamental definition introduced in this thesis is the scaled Pade fraction which exists uniquely. It is shown that scaled Pade fractions satisfy a three-term relationship between elements along an off-diagonal path of the scaled Pade table. This relationship circumvents the problems of the degenerate case - i.e., the problems which plague other relationships such as those upon which the ϵ-algorithm and the qd-algorithm are based.

The three-term relationship is used to develop an $O(n^2)$ algorithm (Algorithm 3) which computes along an off-diagonal path, a finite sequence of successive scaled Pade fractions for the quotient of two power series. In the case that the power series is normal, Algorithm 3 is identical to the one described by Brezinski [*BRZ*76]. In the case that computations progress along the diagonal, Algorithm 3 becomes that of Cabay and Kao [*CAB*83]. Furthermore, it is shown that if the two power series are finite (i.e., polynomials), then Algorithm 3 with computations along one specific off-diagonal path is exactly equivalent to Euclid's extended algorithm for computing the greatest common divisor of two polynomials. However, other off-diagonal paths can be used to compute greatest common divisors, and it remains a subject for future research to determine the optimal one.

By doubling the step at each iteration, the three-term relationship gives rise to an $O(n \log^2 n)$ algorithm (Algorithm 1 or Algorithm 2). The cost complexity assumes the existence of fast methods for polynomial arithmetic. The decision to double the step-size (rather than to triple it, for example) is not accidental. We believe that this choice is optimal (within the constraints placed by fast polynomial arithmetic methods), but a formal proof remains to be obtained.

When applied to polynomials, Algorithm 2 at each iteration routinely produces intermediate polynomial remainder sequences. For this reason, and because Algorithm 2 is simply a fast version of Algorithm 3, Algorithm 2 is truly a fast Euclid's extended algorithm. The PGCD, HGCD and EMGCD algorithms of Moenck [*MOE*73], of Aho, Hopcroft and Ullman [*AHO*74], and of Brent, Gustavson and Yun [*BRE*80], respectively, also compute greatest common divisors with the same cost complexity, but these methods are simply GCD methods. They do not produce intermediate polynomial remainder sequences. In addition, the artificial splitting of polynomials used by these methods to achieve their speed makes them difficult to understand. Finally, these methods are totally recursive, which makes them more costly to implement than Algorithm 2 which is semi-iterative.

For computing scaled Pade fractions for a power series, Brent et al [*BRE*80] have shown that the EMGCD algorithm can be modified, with substantial extra detail, to compute entries along an anti-diagonal path of the Pade table. We have shown that Algorithm 2 and Algorithm 3 can also be used to compute such entries, routinely. From a practical point of view, however, it seems to us that computations along an anti-diagonal path are not as natural as they are along an off-diagonal path. If the choice of the anti-diagonal path is incorrect, computations must be restarted. For off-diagonal computations, n need not be known apriori, and from an application point of view, this may be one of the most important contributions of this thesis.

It is not clear at this time whether or not the fast method, Algorithm 2, is useful in a practical environment. Initial experiments performed by Verheijen [*VER*83] indicate that the fast method, Algorithm 2, outperforms the classical one, Algorithm 3, only when n is greater than approximately 1600. However, as for the fast methods for polynomial multiplication, a hybrid of Algorithm 2 and Algorithm 3 should significantly lower this cross-over point. This remains a subject for future research.

An effective implementation of an off-diagonal algorithm should prove to be a substantial tool in the design of a symbolic and algebraic system. A single routine serves

(1) to obtain rational approximants of power series,

(2) to convert rational funtions from their power series representation,

(3) to compute greatest common divisors of polynomials, and

(4) to solve Hankel and Toeplitz systems of equations.

The scope of this thesis includes those power series with coefficients over a field, only. This restriction is relevant whenever division is required by the off-diagonal algorithms. By choosing to perform pseudo-division, rather than division, the algorithms can be modified so that they are applicable to power series over a Euclidean domain, rather than a field. As for Euclid's extended algorithm, the modified algorithms shall experience exponential growth of coefficients. It is a subject for future research to determine if methods similar to those of Collins [*COL67*] can be used to keep the growth linear. Other plausible methods for extending the results to Euclidean domains, and which need to be investigated, include the Chinese Remainder and Hensel algorithms.

The extension of results to Euclidean domains includes as a special case the multivariate power series. It is of interest to determine how this extension would compare with current methods for solving block Hankel systems, and for constructing Pade approximants for multivariate power series. This remains a subject for future research.

As a final suggestion for future research, the stability of the algorithms for the numerical computation of scaled Pade fractions requires investigation. Some very preliminary results are encouraging.

BIBLIOGRAPHY

[AHO74] Aho, A.V., J.E. Hopcroft and J.D. Ullman., *The Design and Analysis of Computer Algorithms*, Addison-Wesley Mass. 1974.

[AKA73] Akaith, H., "Block Toeplitz matrix inversion", *SIAM J. Appl. Math.* **24** (1973), pp. 234-241.

[BAK75] Baker, G.A.Jr, *Essentials of Pade Approximants*, Academic Press N.Y., 1975.

[BEN80] Bentley J. , D. Hanken and J. Saxe, "A General Method for Solving Divide-and- Conquer Recurrences", *ACM SIGACT NEWS* **3**(1980), pp. 33-44.

[BER68] Berlekamp E. R., *Algebraic Coding Theory*, McGraw-Hill Co. N.Y., 1968.

[BOC22] Bocher, M., *Introduction to High Algebra*, Macmillan Co. N.Y.,1922.

[BOS80] Bose, N.K. and S. Basu,"2-D Matrix: Existence, Non-uniquness and Recursive Computation", *IEEE Trans. AC* (1980), pp. 509-513.

[BRE80] Brent, R., F.G. Gustavson and D.Y.Y. Yun, "Fast Solution of Toeplitz Systems of Equations and Computation of Pade Approximants", *J. of Algorithms* **1** (1980), pp. 259-295.

[BRZ76] Brezinski, C., "Computation of Pade Approximants and Continued Fractions", *J. Comp. Appl. Math.* **2** (1976), pp. 113-123.

[BRO71] Brown, R.S., "On Euclid's Algorithm and the Computation of Polynomial Greatest Common Divisors", *J. ACM* **18**(1971), no.4, pp. 478-504.

[BUL80] Bultheel, A., "Recursive Algorithms for Nonnormal Pade Tables" *SIAM J. Appl. Math.* **39**(1980), No.1, pp. 106-117.

[BUL79] Bultheel, A., "Recursive Algorithms for the Pade Tables: Two Approaches", in *Lecture Notes in Mathematics 765*, Springer-Verlag Berlin, 1979, pp. 210-230.

[CAB79] Cabay, S., CS566: Symbolic and Algebraic Manipulation Class Note, 1979.

[CAB77] Cabay, S. and T. P. Lam, "Congruential Technique for the Exact Solution of Integral Systems of Linear Equation", *ACM Trans. on Math. Software*, 3(1977), no. 4, pp. 386-397.

[CAB83] Cabay, S. and T. T. C. Kao, "The Diagonal Pade Table and the Triangular Decomposition of Hankel Matrices", TR83-10, Spring 1983.

[CLA75] Claessens, G., "A new look at the Pade table and the different methods for computing its elements", *J. of Comp and Appl Math.*, 1(1975), no. 3, pp. 141-152.

[COL66] Collins, G.E., "Polynomial Remainder Sequences and Determinants", *Amer. Math. Mon.* **73**(1966), no.7, pp. 708-712.

[COL67] Collins, G.E., "Subresultants and Reduced Polynomial Remainder Sequences", *J. ACM* **14**(1967), No.1, pp. 128-142.

[COL73] Collins, G.E., "Computer Algebra of Polynomials and Rational Functions", *Amer. Math. Mon.* **80**(1973), pp. 725-754.

[GED79] Geddes, K.O., "Symbolic Computation of Pade Approximants", *ACM Trans. Math. Software*, **5**, pp. 218-233.

[GRA72] Gragg, W.B., "The Pade Table and its Relation to Certain Algorithm of Numerical Anlysis", *SIAM Rev.* **14**(1972), pp. 1-61.

[GRA74] Gragg, W.B., "Matrix Interpretations and Applications of the Continued Fraction Algorithm", *Rocky Mtn. J. Math.*, **4**, pp. 61-637.

[GUS79] Gustavson, F.G. and D.Y.Y. Yun, "Fast Computation of Pade Approximants Toeplitz Systems of Equations via the Extended Euclidean Algorithm", *IBM RC-7551*, 3/9/79.

[GUS79] Gustavson, F.G. and D.Y.Y. Yun, "Fast algorithms for rational Hermite approximation and solution of Toeplitz systems", *IEEE Trans. CS, CAS-26* (1979), No.9, pp. 750-755.

[HOU69] Householder, A.S. and G.W. Stewart, "Bigradients, Hankel Determinants and the Pade Table", in B. Dejon and P. Henrici (eds.), *Constructive Aspects of the Fundamental Theorem of Algebra*, Academic Press, New York, pp. 131-150.

[KAI78] Kailaith, T., A. Vieira and M. Morf, "Inverses of Toeplitz Operators, Innovations and Orthogonal Polynomials", *SIAM Review* **20**, pp. 106-119.

[KNU69] Knuth, D.E., *Seminumeric Algorithms*, Addison-Wesley, 1969.

[LIP76] Lipson, J.D., "Newton's Method: A Greatest Algebraic Algorithm", *Proc. 1976 ACM Symp. on Symbolic and Algebraic Comp.* pp. 260-270.

[LIP81] Lipson, J.D., *Elements of Algebra and Algebraic Computing*, Addison-Wesley, 1981.

[McE78] McEliece, R.J. and J.B. Shearer, "A Property of Euclid's Algorithm and an Application to Pade Approximation", *SIAM J. Appl. Math.* **34**(1978), No.4, pp. 611-617.

[MOS73] Moses, J. and D.Y.Y. Yun, "The EMGCD Algorithm", *Proceeding of ACM annual conf.* (1973), pp. 159-166.

[MOE73] Moenck, R. "Studies in Fast Algebraic Algorithms", *TR-57* University of Toronto, 1973.

[RIS74] Rissanen, J., "Solution of Linear Equations with Hankel and Toeplitz Matrices", *Numer. Math.*, **22**(1974), pp. 361-366.

[TRE64] Trench, W.F., "An Algorithm for the Inversion of Finite Toeplitz Matrices", *J. SIAM* **12**(1964), No.3, pp. 515-522.

[TRE65] Trench, W.F., "An Algorithm for the Inversion of Finite Hankel Matrices", *SIAM J. Appl. Math.*, **13**, pp. 1102-1107.

[VER83] Verheijen, A. *Evaluation of Diagonal Pade Fractions,* M.Sc. Thesis, University of Alberta, 1983, In preparation.

[WYN60] Wynn, P., "The rational approximation of function which are formally defined by a power series expansion", *Math. Comp,.* **14**(1960), pp. 147-186.

[YUN79] Yun, D.Y.Y., "Uniform Bounds for a Class of Algebraic Mappings", *SIAM J. Comp.* **8**(1979), No.3, pp. 348-356.

[YUN74] Yun, D.Y.Y., "The Hensel Lemmma in Algebraic Manipulation", *Mac TR-138,* 1974.

[ZOH69] Zohar, S., "Toeplitz Matrix Inversion: The Algorithm of W.F. Trench", *J. Assoc. Comp. Mach.* **16**(1969), No.4, pp. 792-801.

[ZOH74] Zohar, S., "The solution of a Toeplitz set of linear equations", *J. Assoc. Comp. Mach.* **21**(1974), pp. 272-276.